**Note:** I will not collect this assignment – just do it for your benefit. This is a preparational homework for the final.

**Solve the following problems**

1. Let $X \subset \mathbb{R}^3$ be the union of the spheres of radius 2 centered at $(0,0,0)$ and $(3,0,0)$. Draw $X$ and draw a simplicial complex whose underlying space is homeomorphic to $X$. Compute the Euler characteristic of your complex.

2. Give an explicit homeomorphism between $\mathbb{R}^2$ and the cone $\{(x, y, z) \in \mathbb{R}^3| x^2 + y^2 = z^2, z \geq 0\}$.

3. A space $X$ is *locally compact* at a point $x \in X$ if $x$ has an open neighborhood which itself has a compact neighborhood. We say that $X$ is *locally compact* if it is locally compact at every point.
   
   (a) Prove a compact space is locally compact.
   (b) Prove $\mathbb{R}^n$ is locally compact.
   (c) Prove a punctured surface with boundary is locally compact.
   (d) Is $X := \text{interior}(D^2) \cup \{(1,0)\}$ locally compact?

4. Prove that if $X$ is a compact space, then every sequence $x_1, x_2, x_3, \cdots \in X$ has a cluster point – that is, there is a point $x \in X$ such that every neighborhood of $x$ contains $x_n$ for infinitely many $n$.

5. Consider the subset of $\mathbb{R}^2$ defined by $T_n := \{(x, y) \in \mathbb{R}^2|x = \frac{1}{n} \text{ and } 0 \leq y \leq 1\}$, for any integer $n$, where for the purposes of this problem $T_0 := \{(x, y) \in \mathbb{R}^2|x = 0 \text{ and } 0 \leq y \leq 1\}$. Finally, let $I := \{(x, y)|y = 0 \text{ and } 0 \leq x \leq 1\}$. The topologist’s comb is the union $T := I \cup \bigcup_{i \geq 0} T_i$. For each the following subspaces of $\mathbb{R}^2$, determine which of the following properties it possesses: (i) closed in $\mathbb{R}^2$; (ii) compact; (iii) locally compact; (iv) connected; (v) path-connected; (vi) open in $\mathbb{R}^2$. Justify your response.
   
   (a) The topologist’s comb $T$;
   (b) The topologist’s comb missing a tooth, $M := T - T_0$;
   (c) The broken topologist’s comb, $B := T - \{(0,0)\}$.

6. Let $M := T^2 \# T^2 \# \mathbb{R}^2 \# \mathbb{R}^2$.
   
   (a) What is the Euler characteristic of $M$?
   (b) How is $M$ listed in the classification?
   (c) Give a polygonal disk with gluing scheme such that the quotient space is homeomorphic to $M$. 

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7. Show that there is a 2-sheeted covering $T^2 \to K$ of the Klein bottle by the torus.

8. For some index set $I$, for every $i \in I$ let $A_i \subset \mathbb{R}^n$ be compact. Prove that $\cap_{i \in I} A_i$ is compact.

9. Let $B \subset \mathbb{R}^2$ be a disk, and let $j : \partial B \to \partial B$ be a homeomorphism. Show there is a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H|_{\partial B} = h$.

10. Suppose $G$ is a group for which $x^2 = e$ for each $x \in G$. Show that $ab = ba$ for all $a, b \in G$ (Such groups are called abelian).

11. Let $X$ be a topological space consisting of $n$ points with discrete topology. How many elements are there in the set of homotopy classes of functions from $X$ to

   (a) $\mathbb{R}$

   (b) $\mathbb{R} - \{0\}$

   (c) $\mathbb{R} - \{0, 1\}$

   (d) $\mathbb{R}^2 - \{(0, 0)\}$

12. Let $f : S^n \to S^n$ be a continuous map. Prove that if $f$ does not have fixed points, then $f$ is homotopic to the central symmetry $g(x) = -x$. 
