MAS 3105 Test 3 November 21, 2013

Print your name and your section number and sign below, and read the instructions. Do not open the test until you are told to do so.

Name (printed):	KEY
Section:	
Signature:	

This test has 8 questions on 6 pages. The total number of points is 120.

When the proctor says you may begin then check that you have a complete test.

Put all your answers in the spaces provided on these sheets. The backs of the test sheets are blank and may be used for scratch work. More scratch paper is available on request.

You must show all your work. You must show enough work to indicate how you got your answer. You will lose credit for incorrect statements or incorrect mathematical expressions. Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

You will be graded on what you write in the space provided for your work. Cross out any scratch work, or label it as scratch. If your work is not in the space provided, indicate clearly where we may find it, and label it. Do not give two or more answers for the same problem.

Do not write inside this box.

1		6	
2		7	
3		8	
4			
5			

1. (20 points) Circle "True" at each statement that is always true, and circle "False" at each statement is not always true.

(a) True | False For any matrices A and B such that AB is defined, we have $(AB)^T = B^T A^T$.

(b) True | False For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ ← only if $\vec{u} \perp \vec{v}$

(c) True | False For any subspace V of $\mathbb{R}^n$ the equality $\dim V + \dim V^\perp = n$ holds.

(d) True | False A matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is orthogonal. The columns of A are not unit vectors

(e) True | False If all entries of 3×3 matrix A are 3, then $\det(A) = 3^3$. $\det A = 0$

(f) True | False For any square matrices A and B such that $A+B$ is defined, we have $\det(A+B) = \det(A) + \det(B)$.

(g) True | False The determinant of a $\mathcal{B}$-matrix of a linear transformation does not depend on the choice of a basis $\mathcal{B}$.

(h) True | False A reflection about the xy-plane in $\mathbb{R}^3$ is an orthogonal transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(i) True | False The angle between vectors $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is $\frac{\pi}{6}$. $\cos\alpha = \dfrac{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\left\|\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\| \cdot \left\|\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\|} = \dfrac{1}{2} \Rightarrow \alpha = \dfrac{\pi}{3}$

(j) True | False A change of basis matrix is always invertible.

2. (10 points) Prove that if A is a square invertible symmetric matrix then so is A^{-1}.

For A^{-1} we have:

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

since A is symmetric

So by definition of a symmetric matrix, A^{-1} is symmetric

3. (15 points) Let $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation defined by

$$T(M) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find the matrix $T_{\mathcal{B}}$ of T with respect to standard basis $\mathcal{B}$ of $\mathbb{R}^{2\times 2}$

Let $\mathcal{B} = \left(b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, b_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ be the standard basis for $\mathbb{R}^{2\times 2}$. Then

$$T(b_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = b_3 - b_2$$

$$T(b_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = b_4 - b_1$$

$$T(b_3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = b_1 - b_4$$

$$T(b_4) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = b_2 - b_3$$

Therefore

$$T_{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

(b) Is T an isomorphism? Explain!

No, since the 4th column is (-1) times the first one, so the columns of $T_{\mathcal{B}}$ are linearly dependent.

(c) Find the basis for the image of T.

Since $T(b_4) = -T(b_1)$ and $T(b_3) = -T(b_2)$, and $T(b_1)$ and $T(b_2)$ are not multiples of each other, the matrices $T(b_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T(b_2) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for $\operatorname{Im} T$.

4. (15 points)

(a) Find an orthonormal basis for a subspace $V = \text{span}\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} \right\}$ of $\mathbb{R}^4$.

We apply Gram-Schmidt procedure.

$$\bar{u}_1 = \frac{\bar{v}_1}{\|\bar{v}_1\|} = \frac{1}{\sqrt{1+1+1+1}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

$$\bar{v}_2^{\perp} = \bar{v}_2 - (\bar{v}_2 \cdot \bar{u}_1)\bar{u}_1 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \underbrace{\left(\begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right)}_{-\frac{1}{2}+2+2-\frac{1}{2}} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -\frac{5}{2} \end{bmatrix}$$

$$\bar{u}_2 = \frac{\bar{v}_2^{\perp}}{\|\bar{v}_2^{\perp}\|} = \frac{\frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}{\left\| \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\|} = \frac{1}{\sqrt{1+1+1+1}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Thus, the orthonormal basis for V is $\left\{ \bar{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}$.

(b) Using (a) find the orthogonal projection of $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$ onto V.

$$\text{proj}_V \bar{v}_3 = (\bar{v}_3 \cdot \bar{u}_1)\bar{u}_1 + (\bar{v}_3 \cdot \bar{u}_2)\bar{u}_2 =$$

$$= \left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} =$$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

5. (15 points)

(a) Compute the determinant of the following matrix:

$$A = \begin{bmatrix} 0 & g & 0 & 1 \\ 3 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & h & 0 \end{bmatrix}$$

where g and h denote scalars.

$$\det A = (-1)^{1+2} \cdot g \cdot \det \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & h & 0 \end{bmatrix} + (-1)^{1+4} \cdot 1 \cdot \det \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & h \end{bmatrix} =$$

$$= (-g) \cdot \big(3 \cdot 0 \cdot 0 + (-1) \cdot 1 \cdot 1 + 1 \cdot 2 \cdot h - 1 \cdot 0 \cdot 1 - h \cdot 1 \cdot 3 - 0 \cdot 2 \cdot (-1)\big) +$$

$$(-1) \cdot \big(3 \cdot (-1) h + 2 \cdot 0 \cdot 1 + (-1) \cdot 2 \cdot 0 - 1 \cdot (-1) \cdot (-1) - 0 \cdot 0 \cdot 3 - h \cdot 2 \cdot 2\big) =$$

$$= (-g)(-h-1) + (-1)(-7h-1) = gh + g + 7h + 1$$

(b) Characterize, in terms of g and h, when the matrix in part (a) is invertible.

A is invertible $\Longleftrightarrow$ $gh + g + 7h + 1 \neq 0$.

6. (15 points) Let V be a subspace of $\mathbb{R}^n$. Prove that $V \cap V^\perp = \{\mathbf{0}\}$ (in other words, $\mathbf{0}$ is the only vector that belongs to both V and $V^\perp$).

If $\bar{v} \in V \cap V^\perp$, then $\bar{v} \perp \bar{v} \Rightarrow \bar{v} \cdot \bar{v} = \|\bar{v}\|^2 = 0 \Rightarrow \bar{v} = \bar{0}$.

7. (15 points) Let $T\colon P_1 \to P_1$ be a transformation whose matrix with respect to the standard basis $\mathcal{B} = (1, x)$ of P_1 is $T_{\mathcal{B}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$. Let also $\mathcal{C} = (x-1, x-2)$ be a different basis for P_1.

(a) Find the change of basis matrix $S_{\mathcal{C}\to\mathcal{B}}$.

$$S_{\mathcal{C}\to\mathcal{B}} = \left[[x-1]_{\mathcal{B}} \,\middle|\, [x-2]_{\mathcal{B}} \right] = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

(b) Using (a) find the matrix $T_{\mathcal{C}}$ of T with respect to basis $\mathcal{C}$.

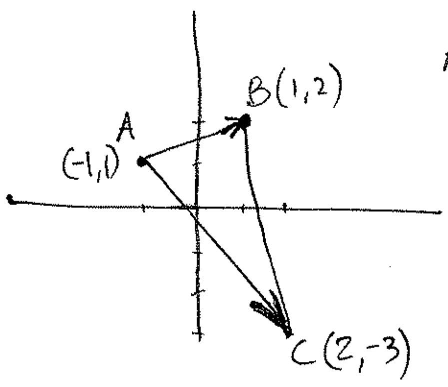
First, we find $S_{\mathcal{B}\to\mathcal{C}} = S_{\mathcal{C}\to\mathcal{B}}^{-1}$:

$$\left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{c|cc} I_2 & 1 & 2 \\ & -1 & -1 \end{array}\right].$$

So $S_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$. Therefore

$$T_{\mathcal{C}} = S_{\mathcal{B}\to\mathcal{C}} T_{\mathcal{B}} \cdot S_{\mathcal{C}\to\mathcal{B}} = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 0 & 1 \end{bmatrix}$$

8. (15 points) Find the area of a triangle whose vertices have coordinates $(-1, 1)$, $(1, 2)$, and $(2, -3)$.



$$\text{Area} = \left|\frac{1}{2}\det\left[\vec{AB} \,\middle|\, \vec{AC}\right]\right| = \frac{1}{2}\left|\det\begin{bmatrix} 1-(-1) & 2-(-1) \\ 2-1 & (-3)-1 \end{bmatrix}\right| =$$

$$= \frac{1}{2}\left|\det\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\right| = \frac{1}{2}\left|(2\cdot(-4) - 1\cdot 3)\right| =$$

$$= \frac{1}{2}\left|(-8-3)\right| = \frac{11}{2}$$