MAS 3105 (Linear Algebra) Test 2 October 24, 2013

Print your name and your section number and sign below, and read the instructions. Do not open the test until you are told to do so.

Name (printed): KEY

Section:

Signature:

This test has 8 questions on 6 pages. The total number of points is 120.

When the proctor says you may begin then check that you have a complete test.

Put all your answers in the spaces provided on these sheets. The backs of the test sheets are blank and may be used for scratch work. More scratch paper is available on request.

You must show all your work. You must show enough work to indicate how you got your answer. You will lose credit for incorrect statements or incorrect mathematical expressions. Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

You will be graded on what you write in the space provided for your work. Cross out any scratch work, or label it as scratch. If your work is not in the space provided, indicate clearly where we may find it, and label it. Do not give two or more answers for the same problem.

Do not write inside this box.

1		6	
2		7	
3		8	
4			
5			

1. (20 points) Circle "True" at each statement that is always true, and circle "False" at each statement is not always true.

(a) $\boxed{\text{True}}\boxed{\text{False}}$ If $T\colon V \to W$ is a linear transformation from a vector space V to a vector space W and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent in V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in W.

(b) $\boxed{\text{True}}\boxed{\text{False}}$ The union of two distinct lines passing through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

(c) $\boxed{\text{True}}\boxed{\text{False}}$ The transformation $D\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ defined by $D(A) = S^{-1}AS$ for some fixed invertible 3×3 matrix S is an isomorphism.

(d) $\boxed{\text{True}}\boxed{\text{False}}$ For each subspace V of $\mathbb{R}^3$ there exists a 3×3 matrix A such that $V = \ker(A)$.

(e) $\boxed{\text{True}}\boxed{\text{False}}$ The dimension of the image of a linear transformation defined by matrix A is equal to the rank of A

(f) $\boxed{\text{True}}\boxed{\text{False}}$ The vector space P_2 is isomorphic to $\mathbb{R}^2$

(g) $\boxed{\text{True}}\boxed{\text{False}}$ For any linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ the equality $\dim\ker(T) + \dim\operatorname{im}(T) = m$ holds.

(h) $\boxed{\text{True}}\boxed{\text{False}}$ The span of polynomials $x + 3$ and $3x - 1$ is equal to P_1

(i) $\boxed{\text{True}}\boxed{\text{False}}$ If vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$ are linearly independent, then $k \leq n$.

(j) $\boxed{\text{True}}\boxed{\text{False}}$ Let $\mathcal{B} = (\mathbf{e}_2, \mathbf{e}_1)$ be a basis of $\mathbb{R}^2$. Then for each $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ we have $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} b \\ a \end{bmatrix}$

2. (10 points) The dimension of the image of a linear transformation $T\colon \mathbb{R}^{110} \to \mathbb{R}^{88}$ is 63.

(a) What is the rank of the standard matrix of T?

The rank of the standard matrix is equal to the dimension of the image of $T = 63$

(b) What is the the dimensions of the kernel of T?

By rank-nullity theorem,

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim(\text{Domain of } T)$$

So

$$\dim \operatorname{Ker} T = \dim(\underbrace{\text{Domain of } T}_{\mathbb{R}^{110}}) - \underbrace{\dim(\operatorname{Im} T)}_{63} =$$

$$= 110 - 63 = 47.$$

3. (15 points) Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$ be a basis for $\mathbb{R}^2$.

(a) Find a vector $\mathbf{v} \in \mathbb{R}^2$ such that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

$$\bar{v} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \cdot [v]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-4 \\ 6-6 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(b) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by the matrix $A = \begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix}$. Find the $\mathcal{B}$-matrix $T_{\mathcal{B}}$ of this linear transformation.

$$T_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \cdot A \cdot \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \frac{1}{1 \cdot 3 - 2 \cdot 2} \cdot \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} =$$

$$= (-1) \cdot \begin{bmatrix} 33-36 & -18+20 \\ -22+18 & 12-10 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 3-4 & 6-6 \\ 4-4 & 8-6 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

4. (15 points) Let $S \colon P_1 \to \mathbb{C}$ be a linear transformation from the vector space P_1 of all polynomials of degree up to 1 to the vector space $\mathbb{C}$ of complex numbers defined by

$$S(f(x)) = f(0) + i \cdot f(2).$$

Prove or disprove that S is an isomorphism.

Since $\dim P_1 = 2 = \dim \mathbb{C}$, it is enough to check that S has trivial kernel.

$S(a + bx) = (a + b \cdot 0) + (a + b \cdot 2)i = a + (2b + a)i = 0 + 0i$

implies $a = 0$ and so $2b + a = 2b = 0 \Rightarrow b = 0$.

Therefore $a + bx \in \operatorname{Ker} S \iff a = b = 0$. So

$\operatorname{Ker} S = \{0\}$ and thus S is an isomorphism.

5. (15 points) The reduced row echelon form of the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & -1 & -1 & 3 & 0 & 0 \\ 0 & 1 & 1 & -3 & 0 & -1 \end{bmatrix}$$

is

$$A' = \begin{bmatrix} 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of A?

rank of A equals to the # of pivot columns $= 3$

(b) Find a basis for the image of A.

Basis for Im A consists of columns in A that correspond to pivot columns in A':

$$\left(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right)$$

(c) Find a basis for the kernel of A.

From $A' = RREF(A)$ we see that x_2, x_3, x_6 are basic vars
x_1, x_4, x_5 are free vars.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1\cdot x_1 + 0 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_1 + 2 \cdot x_4 + (-1) \cdot x_5 \\ 0 \cdot x_1 + 1 \cdot x_4 + 1 \cdot x_5 \\ 0 \cdot x_1 + 1 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_1 + 0 \cdot x_4 + 1 \cdot x_5 \\ 0 \cdot x_1 + 0 \cdot x_4 + 0 \cdot x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, a basis for the Ker A is

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

6. (15 points) Let P_2 denote the space of all polynomials of x of degree at most 2.

(a) Give two different bases for P_2.

$(1, x, x^2)$, $(1+x, x, x^2)$

(b) What is the dimension of P_2?

$\dim P_2 = 3$ as each basis contains 3 elements.

(c) Prove that $f(x) = -x^2 + 2x - 1 \in P_2$ and $g(x) = 2x^2 + x - 3 \in P_2$ are linearly independent.

These 2 polynomials are not multiples of each other, so they are linearly independent.

(d) Let $V = \text{span}\{f(x), g(x)\}$ and $\mathcal{B} = (f(x), g(x))$ be a basis for V. Prove that $h(x) = -6x^2 + 7x - 1$ is in V and compute $[h(x)]_{\mathcal{B}}$.

Let $\mathcal{E} = (1, x, x^2)$ be the standard basis of P_2. Let also $K_{\mathcal{E}}: P_2 \to \mathbb{R}^3$ denote corresponding coordinate transformation. Then $h \in \text{span}\{f, g\} \Leftrightarrow$

$\Leftrightarrow K_{\mathcal{E}}(h) \in \text{span}\{K_{\mathcal{E}}(f), K_{\mathcal{E}}(g)\} \Leftrightarrow$

$$\begin{bmatrix} -1 \\ 7 \\ -6 \end{bmatrix} \in \text{span}\left\{\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}\right\}.$$

To find this out we solve a system of linear equations

$$\left[\begin{array}{cc|c} -1 & -3 & -1 \\ 2 & 1 & 7 \\ -1 & 2 & -6 \end{array}\right] \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 + 2r_1} \left[\begin{array}{cc|c} -1 & -3 & -1 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{array}\right] \xrightarrow{r_3 \to r_3 + r_2} \left[\begin{array}{cc|c} -1 & -3 & -1 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \to -r_1}$$

so $h \in \text{span}\{f, g\}$

$$\to \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \to \frac{r_2}{-5}} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \to r_1 - 3r_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

Therefore, $h(x) = 4f(x) + (-1)g(x)$ and

$$[h(x)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

7. (15 points) Consider the set V of all polynomials $f(x)$ in P_2 that satisfy $f'(1) + f(1) = 0$.

(a) Prove that V is a subspace.

Need to check conditions:

1) $0(x) \in V$: $\quad 0'(1) + 0(1) = 0 + 0 = 0$ ✓

2) $f, g \in V \Rightarrow f'(1) + f(1) = 0,\ g'(1) + g(1) = 0$. Then

$$(f+g)'(1) + (f+g)(1) = \underbrace{f'(1) + f(1) + g'(1) + g(1)}_{\Rightarrow f+g \in V} = 0 + 0 = 0 \checkmark$$

3) $c \in \mathbb{R}, f \in V \Rightarrow f'(1) + f(1) = 0$. Then

$$(cf)'(1) + cf(1) = c(f'(1) + f(1)) = 0 \Rightarrow cf \in V. \checkmark$$

Thus, V is a subspace of P_2

(b) Find the basis for V.

Let $f(x) = a + bx + cx^2$ be an arbitrary polynomial in P_2.

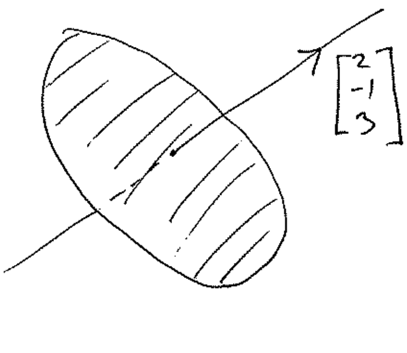
$$f \in V \Leftrightarrow f'(1) + f(1) = (b + 2c) + (a + b + c) = a + 2b + 3c = 0$$

$$\Leftrightarrow f(x) = (-2b - 3c) + bx + cx^2 = b(x-2) + c(x^2 - 3).$$

Since $\{x-2, x^2-3\}$ is linearly independent, the basis for V is

$$\mathcal{B} = (x-2,\ x^2-3).$$

8. (15 points) Find a matrix A whose image is a plane in $\mathbb{R}^3$ defined by the equation $2x - y + 3z = 0$.



Need to find a basis for the solution set of $2x - y + 3z = 0$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \end{array}\right] \xrightarrow{r_1 \to r_1/2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 0 \end{array}\right]$$

$$\underbrace{x}_{\text{basic}} \quad \underbrace{y \quad z}_{\text{free}}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}y - \frac{3}{2}z \\ 1 \cdot y + 0 \cdot z \\ 0 \cdot y + 1 \cdot z \end{bmatrix} = y \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, one can take

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$