Math 3105 | Test 1 | September 24, 2013

Print your name and your section number and sign below, and read the instructions. Do not open the test until you are told to do so.

Name (printed): KEY

Section:

Signature:

This test has 10 questions on 6 pages. The total number of points is 120.

When the proctor says you may begin then check that you have a complete test.

Put all your answers in the spaces provided on these sheets. The backs of the test sheets are blank and may be used for scratch work. More scratch paper is available on request.

You must show all your work. You must show enough work to indicate how you got your answer. You will lose credit for incorrect statements or incorrect mathematical expressions. Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

You will be graded on what you write in the space provided for your work. Cross out any scratch work, or label it as scratch. If your work is not in the space provided, indicate clearly where we may find it, and label it. Do not give two or more answers for the same problem.

Do not write inside this box.

1		6	
2		7	
3		8	
4		9	
5		10	

1. (20 points) Circle "True" at each statement that is always true, and circle "False" at each statement that is not always true.

(a) True False The rank of the augmented matrix of a consistent system of linear equations is equal to the rank of the coefficient matrix of this system.

(b) True False If M is a 95×97 matrix, then the function associated to M cannot be one-to-one.

(c) True False If S is a system of linear equations that has free variables, then S has infinite number of solutions.

(d) True False The function $f(x) = 5x + 5$ from $\mathbb{R}$ to $\mathbb{R}$ is a linear transformation.

(e) True False The reflection $\mathrm{ref}_L\mathbf{x}$ of a vector $\mathbf{x} \in \mathbb{R}^2$ about a line L is equal to $2 \cdot \mathrm{proj}_L\mathbf{x} - \mathbf{x}$.

(f) True False The function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is onto.

(g) True False If in the rref of the augmented matrix of the system S of linear equations the last row contains a leading entry, then S is inconsistent.

(h) True False A square $n \times n$ matrix A is invertible if and only if the rank of A is equal to n.

(i) True False If A and B are both invertible, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

(j) True False If A is $n \times r$ and B is $n \times k$ matrices, then AB is $r \times k$ matrix.

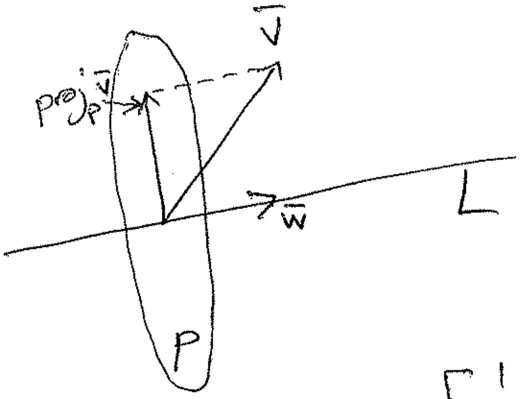
2. (10 points) Use elementary row operations to put the following matrix into reduced row echelon form. Clearly identify each row operation you perform, and perform only one operation at a time.

$$\begin{bmatrix} 4 & 2 & -2 & -1 \\ -1 & 0 & -1 & 1 \\ 2 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} -1 & 0 & -1 & 1 \\ 4 & 2 & -2 & -1 \\ 2 & 0 & 2 & 1 \end{bmatrix} \xrightarrow[r_3 \to r_3 + 2r_1]{r_2 \to r_2 + 4r_1} \begin{bmatrix} -1 & 0 & -1 & 1 \\ 0 & 2 & -6 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{r_2 \to r_2 - r_3}$$

$$\to \begin{bmatrix} -1 & 0 & -1 & 1 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[r_3 \to r_3/3,\ r_1 \to r_1(-1)]{r_2 \to r_2/2} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \to r_1 + r_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. (15 points) Let L be a line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Find the projection of a vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ on the plane P orthogonal to line L.



$$\mathrm{proj}_P \bar{v} = \bar{v} - \mathrm{proj}_L \bar{v} = \bar{v} - \frac{\bar{v}\cdot\bar{w}}{\bar{w}\cdot\bar{w}}\cdot\bar{w} =$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}.$$

4. (8 points) Suppose that A, B, C, D and E are matrices with the following dimensions:
A is 4×5, B is 4×5, C is 4×2, D is 5×2, E is 5×4.
Are the following expressions defined? Write "DEFINED" or "NOT DEFINED" by each to indicate your answer. If an expression is defined, then give its dimensions.

(a) AD+C

$A_{4\times5}\, D_{5\times2} + C_{4\times2} = (AD)_{4\times2} + C_{4\times2} = (AD+C)_{4\times2}$ — defined, 4×2

(b) E(A+B)

$E_{5\times4}\,(A_{4\times5} + B_{4\times5}) = E_{5\times4}\cdot(A+B)_{4\times5} = (E(A+B))_{5\times5}$ — defined 5×5

(c) (EC)D

$(E_{5\times4}\, C_{4\times2})\, D_{5\times2} = (EC)_{5\times2}\cdot D_{5\times2}$ — NOT DEFINED

(d) AB

$A_{4\times5}\cdot B_{4\times5}$ — NOT DEFINED

5. (12 points) Find the solution set to the following system of equations:

$$3x_1 - x_2 + 9x_3 = 0$$
$$-2x_1 + x_2 + 3x_3 = 1$$

The augmented matrix of a system is:

$$\left[\begin{array}{ccc|c} 3 & -1 & 9 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right] \xrightarrow{r_1 \to r_1 + r_2} \left[\begin{array}{ccc|c} 1 & 0 & 12 & 1 \\ -2 & 1 & 3 & 1 \end{array}\right] \xrightarrow{r_2 \to r_2 + 2r_1}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 12 & 1 \\ 0 & 1 & 27 & 3 \end{array}\right].$$

basic vars x_1, x_2 ; free var x_3

The general solution is:

$$\boxed{\begin{array}{l} x_1 = 1 - 12x_3 \\ x_2 = 3 - 27x_3 \\ x_3 = x_3 \end{array}}$$

6. (5 points) Find two different row echelon forms of $\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \underbrace{\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}}_{\text{REF 1}} \xrightarrow{r_2 \to r_2/2} \underbrace{\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}}_{\text{REF2}}$$

7. (15 points)

(a) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 + r_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -1 & +2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{r_3 \to r_3 + r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] -$$

$$\xrightarrow[r_1 \to r_1 + r_3]{r_2 \to r_2 + r_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_1 \to r_1 - 2r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & -1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right].$$

Thus, $A^{-1} = \begin{bmatrix} -1 & -3 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(b) Use your answer from part (a) to solve the equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\bar{x} = A^{-1}\bar{b} = \begin{bmatrix} -1 & -3 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

8. (10 points) Find a nonzero and nonidentity 2×2 matrix A such that $A^2 = A$.

We can take a matrix corresponding to a projection onto the x-axis:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$.

There are many others.

9. (10 points) Let $D = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 0 & 1 \end{bmatrix}$.

(a) Find the rank of D.

$$\begin{bmatrix} 1 & -2 & 1 \\ -4 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \to r_2 + 4r_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -8 & 5 \end{bmatrix}$$

Since there are 2 pivot columns, the rank is equal to 2.

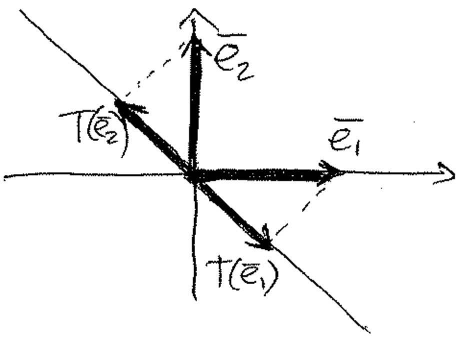
(b) Is the function that is associated to D an onto function? Explain how you know.

Yes, since $\operatorname{rank} D = 2 = \#$ of rows.

(c) Is the function that is associated to D a one-to-one function? Explain how you know.

No, since $\operatorname{rank} D = 2 \neq 3 = \#$ of columns.

10. (15 points) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation projecting all vectors in $\mathbb{R}^2$ orthogonally onto a line parallel to the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (i.e. a line $y = -x$). Find the matrix that corresponds to F. Explain how you get it!



We have that the matrix of F is

$$\left[F(\bar{e}_1) \,\middle|\, F(\bar{e}_2) \right] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$