MAS 3105 (Linear Algebra) Your name: KEY

Quiz 5

March 25, 2014

1. Let V be a vector space and let $\mathcal{B}, \mathcal{C}$ be two bases for V. Define the change of basis matrix $S_{\mathcal{B}\to\mathcal{C}}$.

The change of basis matrix $S_{B\to C}$ is the standard matrix of a linear transformation
$K_C \circ K_B^{-1} : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{tikzcd} & \mathbb{R}^n \arrow[d, "S_{B\to C}"] \\ V \arrow[ur, "K_B"] \arrow[r, "K_C"'] & \mathbb{R}^n \end{tikzcd}$$

2. Let $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation defined by

$$T(M) = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} M$$

(a) Find the matrix $T_{\mathcal{B}}$ of T with respect to a basis

$$\mathcal{B} = \left\{ \underset{u_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{u_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}.$$

$$\left.\begin{aligned}
T(u_1) &= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \\
T(u_2) &= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \\
T(u_3) &= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \\
T(u_4) &= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}
\end{aligned}\right\} \Rightarrow T_B = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

(b) Determine whether T is an isomorphism. Explain your answer. If T is not an isomorphism, find a basis for the image of T.

$\operatorname{rank} T_B = 2$ since column 3 = column 1, column 4 = column 2, columns 1 an 2 are not multiples of each other.

Since $\operatorname{rank} T_B < 4$, T is not an isomorphism.

Basis for the image of T is

$$\left(K_B^{-1}\left(\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}\right), K_B^{-1}\left(\begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}\right) \right) = \left(\begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \right)$$

basis for $\operatorname{Im} T_B$.