Linear Algebra

Math 3105 **Test 2** **November 7, 2012**

Print your name and your section number and sign below, and read the instructions. Do not open the test until you are told to do so.

Name (printed): KEY

Section:

Signature:

This test has 11 questions on 8 pages. The total number of points is 100.

When the proctor says you may begin then check that you have a complete test.

Put all your answers in the spaces provided on these sheets. The backs of the test sheets are blank and may be used for scratch work. More scratch paper is available on request.

You must show all your work. You must show enough work to indicate how you got your answer. You will lose credit for incorrect statements or incorrect mathematical expressions. Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

You will be graded on what you write in the space provided for your work. Cross out any scratch work, or label it as scratch. If your work is not in the space provided, indicate clearly where we may find it, and label it. Do not give two or more answers for the same problem.

Do not write inside this box.

1		6		11	
2		7			
3		8			
4		9			
5		10			

1. (20 points) Circle "True" at each statement that is always true, and circle "False" at each statement is not always true.

(a) True False For any matrices A and B such that AB is defined, we have $(AB)^T = A^T B^T$ is always symmetric.

(b) True False The vector space P_1 is isomorphic to $\mathbb{C}$

(c) True False The set of points in $\mathbb{R}^2$ with integer coordinates is a subspace of $\mathbb{R}^2$

(d) True False The transformation $D(f) = f'$ from C^∞ to C^∞ is an isomorphism.

(e) True False For each subspace V of $\mathbb{R}^3$ there exists a 3×3 matrix A such that $V = Im(A)$.

(f) True False For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have $||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u}|| + ||\mathbf{v}||$

(g) True False For any subspace V of $\mathbb{R}^n$ the equality $\dim V + \dim V^\perp = n$ holds.

(h) True False A matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is orthogonal. (the columns are orthogonal, but not orthonormal)

(i) True False If vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ span $\mathbb{R}^n$, then $k \geq n$.

(j) True False If $\mathcal{B} = (f, g)$ and $\mathcal{U} = (f, f + g)$ are two bases of a linear space V, then the change of basis matrix $S_{\mathcal{B} \to \mathcal{U}}$ from $\mathcal{B}$ to $\mathcal{U}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

2. (5 points) Let $\mathcal{B} = \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \end{bmatrix} \right)$ be a basis for $\mathbb{R}^2$. Find coordinates of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with respect $\mathcal{B}$.

We have to find $c_1, c_2 \in \mathbb{R}$ s.t. $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 9 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & 9 & 1 \\ 4 & 1 & 1 \end{array}\right] \xrightarrow{-\mathrm{II}} \left[\begin{array}{cc|c} -1 & 8 & 0 \\ 4 & 1 & 1 \end{array}\right] \xrightarrow{*(-1)} \left[\begin{array}{cc|c} 1 & -8 & 0 \\ 4 & 1 & 1 \end{array}\right] \xrightarrow{-4\cdot\mathrm{I}} \left[\begin{array}{cc|c} 1 & -8 & 0 \\ 0 & 33 & 1 \end{array}\right] \xrightarrow{*\frac{1}{33}}$$

$$\to \left[\begin{array}{cc|c} 1 & -8 & 0 \\ 0 & 1 & 1/33 \end{array}\right] \xrightarrow{+8\cdot\mathrm{II}} \left[\begin{array}{cc|c} 1 & 0 & 8/33 \\ 0 & 1 & 1/33 \end{array}\right].$$

Thus, $\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 8/33 \\ 1/33 \end{bmatrix}$

3. (10 points) The reduced row echelon form of the matrix

$$A = \begin{bmatrix} -1 & -2 & 1 & -2 & -9 & -1 \\ -2 & -4 & 2 & 1 & 2 & -1 \\ 3 & 6 & -3 & 0 & 3 & 1 \\ 2 & 4 & -2 & 2 & 10 & 1 \\ 1 & 2 & -1 & -1 & -3 & 1 \end{bmatrix}$$

is

$$A' = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of A?

There are 3 pivot columns in A', so $\operatorname{rank} A = 3$

(b) Find a basis for the image of A.

The basis for $\operatorname{Im}(A)$ consists of columns of A that correspond to pivot columns in A'. In this case:

$$\text{Basis for } \operatorname{Im}(A) = \left(\begin{bmatrix} -1 \\ -2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

(c) Find a basis for the kernel of A.

We have $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \in \operatorname{Ker}(A) \Leftrightarrow$

$$\bar{x} = \begin{bmatrix} -2x_2 + 1\cdot x_3 - 1\cdot x_5 \\ 1\cdot x_2 + 0\cdot x_3 + 0x_5 \\ 0\cdot x_2 + 1\cdot x_3 + 0\cdot x_5 \\ 0\cdot x_2 + 0\cdot x_3 - 4\cdot x_5 \\ 0\cdot x_2 + 0\cdot x_3 + 1\cdot x_5 \\ 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_5 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, basis for $\operatorname{Ker}(A)$ is

$$\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right)$$

4. (10 points) Suppose that S is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, with

$$S\left(\begin{bmatrix}1\\3\\-2\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix}, S\left(\begin{bmatrix}6\\-3\\0\end{bmatrix}\right)=\begin{bmatrix}3\\0\\0\end{bmatrix}.$$

Find a nonzero vector that is in the kernel of S.

Since for $\bar{v}_1=\begin{bmatrix}1\\3\\-2\end{bmatrix}$ and $\bar{v}_2=\begin{bmatrix}6\\-3\\0\end{bmatrix}$ we have

$3S(\bar{v}_1)-S(\bar{v}_2)=\bar{0}$, we get

$S(3\bar{v}_1-\bar{v}_2)=\bar{0}$, So $3\bar{v}_1-\bar{v}_2=\begin{bmatrix}-3\\12\\-6\end{bmatrix}$ is a nonzero vector in $\operatorname{Ker} S$

5. (10 points) Let P_2 denote the space of all polynomials of x of degree at most 2.

(a) Give a basis for P_2?

$\mathcal{B}=(1, x, x^2)$

(b) What is the dimension of P_2?

3, since P_2 has a basis of 3 elements

(c) Prove that $f(x)=x^2-x+1\in P_2$ and $g(x)=x^2+x+1\in P_2$ are linearly independent.

Let $L_{\mathcal{B}}$ be coordinate transformation from P_2 to $\mathbb{R}^3$.

$\{f(x), g(x)\}$ is lin. indep. $\Leftrightarrow \{L_{\mathcal{B}}(f), L_{\mathcal{B}}(g)\}$ is lin. indep. in $\mathbb{R}^3$

$$\begin{bmatrix}1&1\\-1&1\\1&1\end{bmatrix}\begin{matrix}\\+I\\-I\end{matrix}\rightarrow\begin{bmatrix}1&1\\0&2\\0&0\end{bmatrix}.$$

$L_{\mathcal{B}}(f)$ $L_{\mathcal{B}}(g)$

So $\operatorname{rank}\left(\left[L_{\mathcal{B}}(f)\,\middle|\,L_{\mathcal{B}}(g)\right]\right)=2$ and $\{f,g\}$ is linearly independent

(d) Express $h(x)=x^2-5x+1\in P_2$ as a linear combination of $f(x)$ and $g(x)$.

Again, we use coordinate transformation:

$$\left[\begin{array}{cc|c}1&1&1\\-1&1&-5\\1&1&1\end{array}\right]\begin{matrix}\\+I\\-I\end{matrix}\rightarrow\left[\begin{array}{cc|c}1&1&1\\0&2&-4\\0&0&0\end{array}\right]\begin{matrix}\\ *\tfrac{1}{2}\\ \\ \end{matrix}\rightarrow\left[\begin{array}{cc|c}1&1&1\\0&1&-2\\0&0&0\end{array}\right]\begin{matrix}-II\\ \\ \\ \end{matrix}\rightarrow\left[\begin{array}{cc|c}1&0&3\\0&1&-2\\0&0&0\end{array}\right].$$

Therefore, $h(x)=3f(x)-2g(x)$.

6 (10 points) Consider the set V of all polynomials $f(x)$ in P_2 that satisfy $f'(1) = 0$.

(a) Prove that V is a subspace.

Need to check 3 conditions

1) 0 polynomial in V: true, since $(0(x))' = 0$ and $0(1) = 0$

2) Let $f, g \in V$. Then $(f+g)'(1) = f'(1) + g'(1) = 0 + 0 = 0$

So $f + g \in V$

3) Let $f \in V$ and $k \in \mathbb{R}$. Then

$(kf)'(1) = k f'(1) = k \cdot 0 = 0$. So $kf \in V$.

Thus, V is a subspace

(b) Find the basis for V.

Let $f(x) = ax^2 + bx + c$ be arbitrary polynomial in P_2.

$f \in V \Leftrightarrow f'(1) = 2a \cdot 1 + b = 0 \Leftrightarrow b = -2a$. So

$f \in V \Leftrightarrow f(x) = ax^2 - 2ax + c = a(x^2 - 2x) + c \cdot 1$.

Therefore, $\operatorname{span}(1, x^2 - 2x) = V$.

Also, $\{1, x^2 - 2x\}$ is lin. independent as none of these polynomials is a multiple of the other.

Thus $(1, x^2 - 2x)$ is a basis for V.

7 (5 points) Prove that for any matrix A the matrix $A^T A$ is symmetric.

We have for $B = A^T A$

$B^T = (A^T A)^T = A^T \cdot (A^T)^T = A^T \cdot A = B$.

Thus, $A^T A = B$ is symmetric.

8. (15 points) Let $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation defined by

$$T(M) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} M - M \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(a) Find the matrix $T_{\mathcal{B}}$ of T with respect to standard basis $\mathcal{B}$ of $\mathbb{R}^{2\times 2}$

We first find $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$

$$= \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} - \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & c \end{bmatrix}.$$

Therefore,

$$T_{\mathcal{B}} = \left[\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{B}} \middle| \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{B}} \middle| \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{\mathcal{B}} \middle| \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{\mathcal{B}} \right] =$$

$$= \left[\left[\begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix}\right]_{\mathcal{B}} \middle| \left[\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right]_{\mathcal{B}} \middle| \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]_{\mathcal{B}} \middle| \left[\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}\right]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) Is T an isomorphism? Explain!

No, as the rank of $T_{\mathcal{B}}$ is less than 4.

(c) Find the basis for the image of T.

It is obvious that columns 2 and 4 in $T_{\mathcal{B}}$ are redundant, and columns 1 and 3 are linearly indep.

So

$$\text{basis}(\operatorname{Im} T) = L_{\mathcal{B}}^{-1}(\text{basis}(\operatorname{Im} T_{\mathcal{B}})) = L_{\mathcal{B}}^{-1}\left(\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

9. (10 points)

(a) Find the orthonormal basis of a subspace V of $\mathbb{R}^4$ spanned by vectors $\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\9\\-5\\3\end{bmatrix}$

We use Gram-Schmidt orthogonalization process.

$$\bar{u}_1 = \frac{\bar{v}_1}{\|\bar{v}_1\|} = \frac{1}{\sqrt{1^2+1^2+1^2+1^2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}.$$

$$\bar{v}_2^{\perp} = \bar{v}_2 - (\bar{v}_2\cdot\bar{u}_1)\bar{u}_1 = \begin{bmatrix}1\\9\\-5\\3\end{bmatrix} - \left(\begin{bmatrix}1\\9\\-5\\3\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\9\\-5\\3\end{bmatrix} - \frac{1}{4}\cdot(1+9-5+3)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\7\\-7\\1\end{bmatrix}$$

$$\bar{u}_2 = \frac{\bar{v}_2^{\perp}}{\|\bar{v}_2^{\perp}\|} = \frac{1}{\sqrt{1^2+7^2+7^2+1^2}}\begin{bmatrix}-1\\7\\-7\\1\end{bmatrix} = \begin{bmatrix}-1/10\\7/10\\-7/10\\1/10\end{bmatrix}.$$

Then $(\bar{u}_1, \bar{u}_2)$ is an orthonormal basis for V.

(b) Using (a) find the orthogonal projection of $\mathbf{v}_3 = \begin{bmatrix}1\\2\\3\\-2\end{bmatrix}$ onto V

$$\mathrm{Proj}_V\,\bar{v}_3 = (\bar{v}_3\cdot\bar{u}_1)\bar{u}_1 + (\bar{v}_3\cdot\bar{u}_2)\bar{u}_2 =$$

$$= \left(\begin{bmatrix}1\\2\\3\\-2\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \left(\begin{bmatrix}1\\2\\3\\-2\end{bmatrix}\cdot\frac{1}{10}\begin{bmatrix}-1\\7\\-7\\1\end{bmatrix}\right)\frac{1}{10}\begin{bmatrix}-1\\7\\-7\\1\end{bmatrix} =$$

$$= \frac{1}{4}(1+2+3-2)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{100}\cdot(-1+14-21-2)\begin{bmatrix}-1\\7\\-7\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1}{10}\begin{bmatrix}-1\\7\\-7\\1\end{bmatrix} = \begin{bmatrix}1.1\\0.3\\1.7\\0.9\end{bmatrix}$$

10. (10 points) Let $\mathcal{B} = \left(\overset{\bar{b}_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \overset{\bar{b}_2}{\begin{bmatrix}0\\1\end{bmatrix}} \right)$ and $\mathcal{U} = \left(\overset{\bar{u}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \overset{\bar{u}_2}{\begin{bmatrix}-1\\1\end{bmatrix}} \right)$ be two bases of $\mathbb{R}^2$. Suppose a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ has a $\mathcal{B}$-matrix $T_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$. Find the matrix $T_{\mathcal{U}}$ of T with respect to basis $\mathcal{U}$.

First, we find $S_{\mathcal{U}\to\mathcal{B}}$. We need to express $\bar{u}_1, \bar{u}_2$ in terms of $\bar{b}_1, \bar{b}_2$:

$$\begin{array}{c} \left[\begin{array}{cc|c:c} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right] \\ \bar{b}_1 \;\; \bar{b}_2 \;\; \bar{u}_1 \;\; \bar{u}_2 \end{array} \xrightarrow{-\mathrm{I}} \left[\begin{array}{cc|c:c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]. \text{ Thus } [\bar{u}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, [\bar{u}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and $S_{\mathcal{U}\to\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

Now we find $S_{\mathcal{B}\to\mathcal{U}}$ as $S_{\mathcal{U}\to\mathcal{B}}^{-1}$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{+\mathrm{I}} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{+\mathrm{II}} \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]. \text{ So } S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus, $T_{\mathcal{U}} = S_{\mathcal{B}\to\mathcal{U}} T_{\mathcal{B}} S_{\mathcal{U}\to\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} =$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

11. (5 points) A linear transformation $T\colon \mathbb{R}^{100} \to \mathbb{R}^{105}$ is given by a matrix of rank 85. What are the dimensions of the image and the kernel of T?

By rank-nullity theorem:

$$\underbrace{\dim(\text{domain of } (T))}_{100} = \dim(\mathrm{Ker}(T)) + \underbrace{\dim(\mathrm{Im}(T))}_{85}$$

equals to the rank of the matrix

Thus, $\dim(\mathrm{Im}\,T) = 85$

$\dim(\mathrm{Ker}\,T) = 100 - 85 = 15$