Math 3105 Test 1 September 26, 2012

Print your name and your section number and sign below, and read the instructions. Do not open the test until you are told to do so.

Name (printed): KEY

Section:

Signature:

This test has 11 questions on 7 pages. The total number of points is 100.

When the proctor says you may begin then check that you have a complete test.

Put all your answers in the spaces provided on these sheets. The backs of the test sheets are blank and may be used for scratch work. More scratch paper is available on request.

You must show all your work. You must show enough work to indicate how you got your answer. You will lose credit for incorrect statements or incorrect mathematical expressions. Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

You will be graded on what you write in the space provided for your work. Cross out any scratch work, or label it as scratch. If your work is not in the space provided, indicate clearly where we may find it, and label it. Do not give two or more answers for the same problem.

Do not write inside this box.

1		6		11	
2		7			
3		8			
4		9			
5		10			

1. (20 points) Circle "True" at each statement that is always true, and circle "False" at each statement is not always true.

(a) True | False The rank of the coefficient matrix of a system S is equal to the number of free variables in S.

should be basic for "True".

(b) True | False If M is a 95×97 matrix, then the function associated to M cannot be a one-to-one correspondence.

(c) True | False If S is a system of linear equations that has no free variables, then S must be consistent.

Last column in augmented matrix still can be pivot

(d) True | False The function $f(x) = x^2$ from $[0,1]$ to $[0,1]$ is one-to-one

(e) True | False If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$. $(AB)^{-1} = B^{-1}A^{-1}$

(f) True | False If A and B are both invertible, then $A + B$ is invertible as well.

invertible $I_n + (-I_n) = 0$ not invertible

(g) True | False If A is $m \times n$ and B is $n \times k$ matrices, then AB is $m \times k$ matrix.

(h) True | False If A is an invertible matrix and c is a nonzero real number, then $(cA)^{-1} = cA^{-1}$.

$(cA)^{-1} = c^{-1}A^{-1}$

(i) True | False If A is any $n \times n$ invertible matrix, then $\text{rref}(A) = I_n$.

(j) True | False The function $f(x) = 2x + 3$ from $\mathbb{R}$ to $\mathbb{R}$ is a linear transformation.

$f(0) = 3 \neq 0$. But for each linear transformation $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$.

2. (7 points) Use elementary row operations to put the following matrix into reduced row echelon form. Clearly identify each row operation you perform, and perform only one operation at a time.

$$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 3 & 5 & 2 \\ -2 & -1 & -3 & 0 \end{bmatrix} \begin{matrix} \\ -4\cdot I \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -10 \\ -2 & -1 & -3 & 0 \end{bmatrix} \begin{matrix} \\ \\ +2\cdot I \end{matrix}$$

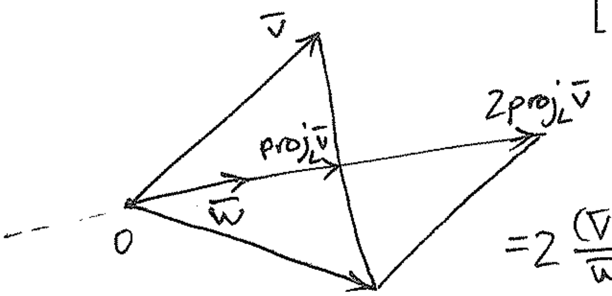
$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -10 \\ 0 & 1 & -1 & 6 \end{bmatrix} \begin{matrix} \\ \\ +II \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -10 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{matrix} \\ \\ *(-\frac{1}{4}) \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ +10\cdot III \\ \\ \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -3\cdot III \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ *(-1) \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -II \\ \\ \\ \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow \text{this is rref}(A)$$

3. (8 points) Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Find the reflection of a vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line L.



We have

$$ref_L(\bar{v}) = 2\, proj_L(\bar{v}) - \bar{v} =$$

$$= 2\,\frac{(\bar{v}\cdot\bar{w})}{\bar{w}\cdot\bar{w}}\cdot\bar{w} - \bar{v} = 2\cdot\frac{2\cdot 1 + 1\cdot 1 + 2\cdot 1}{2^2+1^2+2^2}\cdot\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \frac{10}{9}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{20}{9} - 1 \\ \frac{10}{9} - 1 \\ \frac{20}{9} - 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 11 \\ 1 \\ 11 \end{bmatrix}$$

4. (8 points) Suppose that A, B, C, D and E are matrices with the following dimensions:
A is 4×5, B is 4×5, C is 5×2, D is 4×2, E is 5×4.
Are the following expressions defined? Write "DEFINED" or "NOT DEFINED" by each to indicate your answer. If an expression is defined, then give its dimensions.

(a) BA $\quad {}_4\overset{5}{\boxed{B}} \cdot {}_4\overset{5}{\boxed{A}}$ $\quad$ Not Defined

(b) AC+D $\quad {}_4\overset{5}{\boxed{A}} \cdot {}_5\overset{2}{\boxed{C}} + {}_4\overset{2}{\boxed{D}} = {}_4\overset{2}{\boxed{AC}} + {}_4\overset{2}{\boxed{D}} = {}_4\overset{2}{\boxed{AC+D}}$. $\quad$ So it is Defined and is 4×2 matrix

(c) E(A+B) $\quad {}_5\overset{4}{\boxed{E}}\left({}_4\overset{5}{\boxed{A}} + {}_4\overset{5}{\boxed{B}}\right) = {}_5\overset{4}{\boxed{E}} \cdot {}_4\overset{5}{\boxed{A+B}} = {}_5\overset{5}{\boxed{E(A+B)}}$. $\quad$ Defined 5×5

(d) (ED)C

$\left({}_5\overset{4}{\boxed{E}} \cdot {}_4\overset{2}{\boxed{D}}\right) {}_5\overset{2}{\boxed{C}} = {}_5\overset{2}{\boxed{ED}} \cdot {}_5\overset{2}{\boxed{C}}$ $\quad$ Not defined

5. (9 points) Find the solution set to the following system of equations:

$$5x_1 - 2x_2 + 6x_3 = 0$$
$$-2x_1 + x_2 + 3x_3 = 1$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right] \xrightarrow{+2\cdot \mathrm{II}} \left[\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ -2 & 1 & 3 & 1 \end{array}\right]_{+2\cdot \mathrm{I}} \rightarrow$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array}\right].$$

$$\underbrace{x_1 \quad x_2}_{\text{basic}} \quad \underbrace{x_3}_{\text{free}}$$

Thus, the solution set is

$$\boxed{\begin{aligned} x_1 &= 2 - 12x_3 \\ x_2 &= 5 - 27x_3 \\ x_3 &= x_3 \text{ - anything} \end{aligned}}$$

or

$$x_1 = 2 - 12t, \quad x_2 = 5 - 27t, \quad x_3 = t, \qquad t \in \mathbb{R} \text{ - arbitrary}$$

6. (5 points) Find two different row echelon forms of $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}_{-2\cdot \mathrm{I}} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$ — one of the row echelon forms

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^{-3\cdot \mathrm{II}} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ — another row echelon form which is, at the same time, reduced row echelon form.

7. (10 points)

(a) Let $A = \begin{bmatrix} 1 & 2 \\ 6 & 2 \end{bmatrix}$. Find A^{-1}.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{array}\right] -6\cdot\mathrm{I} \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -10 & -6 & 1 \end{array}\right] * \left(-\tfrac{1}{10}\right) \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{5} & -\frac{1}{10} \end{array}\right] -2\cdot\mathrm{II}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1-\frac{6}{5} & 0+\frac{2}{10} \\ 0 & 1 & \frac{3}{5} & -\frac{1}{10} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{5} & \frac{1}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{1}{10} \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{10} \end{bmatrix}$

(b) Use your answer from part (a) to solve the equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$, where $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$A\bar{x} = \bar{b} \implies \bar{x} = A^{-1}\bar{b} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} + \frac{4}{5} \\ \frac{9}{5} - \frac{4}{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \end{bmatrix}$$

8. (6 points) If $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix}$, then find $(AB)^{-1}$.

$$(AB)^{-1} = B^{-1} \cdot A^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2\cdot3+5\cdot1 & 2\cdot2+5\cdot3 \\ 3\cdot3-2\cdot1 & 3\cdot2-2\cdot3 \end{bmatrix} = \begin{bmatrix} 11 & 19 \\ 7 & 0 \end{bmatrix}$$

9. (10 points) Let $D = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 0 \end{bmatrix}$.

(a) Find the rank of D.

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{3}{4}\cdot II} \begin{bmatrix} 0 & 2 & 1 \\ 4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

So rank $D = 2$ since there are 2 pivot columns in ref (D).

(b) Is the function that is associated to D an onto function? Explain how you know.

Yes, since rank(D) = number of rows.

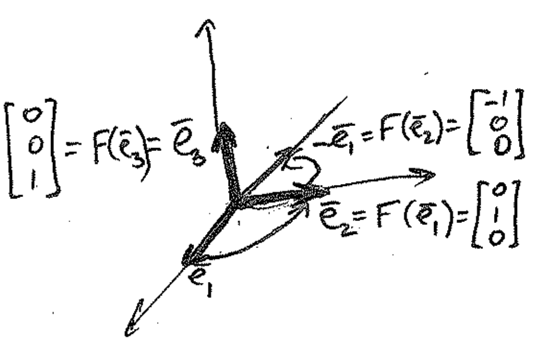
(c) Show that the function represented by the matrix D is not one-to-one explicitly, by finding two different vectors $\mathbf{v}, \mathbf{w}$ such that $D\mathbf{v} = D\mathbf{w}$.

As $\bar{v}$ we can take $\bar{0}$. Then we need to find $\bar{w} \neq \bar{0}$ s.t. $D\bar{w} = D\bar{v} = D \cdot \bar{0} = \bar{0}$. We continue row reduction from (a):

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} *\frac{1}{4} \\ *\frac{1}{2} \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

So general solution of a homogeneous system is $\boxed{\begin{cases} x_1 = 0 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 = \text{anything} \end{cases}}$. In particular, setting $x_3 = 2 \neq 0$, gives $\bar{w} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$.

10. (7 points) $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, which rotates all vectors about z axis by the angle $\pi/2$ counterclockwise if one looks from the top of the z axis. Find the matrix that corresponds to F. Explain how you get it!



The matrix corresponding to F can be computed as

$$\left[F(\bar{e}_1) \middle| F(\bar{e}_2) \middle| F(\bar{e}_3) \right] \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

11. (10 points) If $\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix}$, then find a, b, c, d.

We have

$$\begin{cases} 1\cdot a + 1\cdot b + 0\cdot c + 0\cdot d = 4 \\ 0\cdot a + 0\cdot b + 1\cdot c + 1\cdot d = 6 \\ 0\cdot a + 0\cdot b + 1\cdot c - 1\cdot d = 10 \\ 1\cdot a - 1\cdot b + 0 c + 0 d = 2 \end{cases}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & -1 & 10 \\ 1 & -1 & 0 & 0 & 2 \end{array}\right]\begin{array}{l} \\ \\ -\mathrm{II} \\ -\mathrm{I} \end{array} \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & -2 & 0 & 0 & -2 \end{array}\right] * \left(-\tfrac{1}{2}\right)$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right] * \left(-\tfrac{1}{2}\right) \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\begin{array}{l} -\mathrm{II} \\ \\ -\mathrm{IV} \\ \\ \end{array} \to$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right].$$

Thus $\boxed{a = 3, \quad b = 1, \quad c = 8, \quad d = -2}$