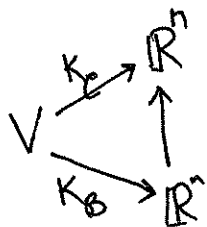


## Quiz 6

November 5, 2013

1. Let  $V$  be a vector space and let  $B, C$  be two bases for  $V$ . Define the change of basis matrix  $S_{B \rightarrow C}$ .



Change of basis matrix  $S_{B \rightarrow C}$  is the standard matrix of a linear transformation  $K_C \circ K_B^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , where  $K_B$  and  $K_C$  are coordinate isomorphisms.

2. Let  $T: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$  be a linear transformation defined by

$$T(M) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} M$$

- (a) Find the matrix  $T_B$  of  $T$  with respect to a basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

$b_1 \qquad b_2 \qquad b_3 \qquad b_4$

$$\left. \begin{aligned} T(b_1) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \\ T(b_2) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \\ T(b_3) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \\ T(b_4) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \end{aligned} \right\} \Rightarrow T_B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

- (b) Determine whether  $T$  is an isomorphism. Explain your answer. If  $T$  is not an isomorphism, find a basis for the image of  $T$ .

Since  $\text{rank } T_B = 2$ ,  $T$  is not an isomorphism. If  $\bar{v}_i$  denotes the  $i$ -th column of  $T_B$ , then  $\bar{v}_3 = \bar{v}_1$  and  $\bar{v}_4 = \bar{v}_2$ . Since  $\bar{v}_1$  and  $\bar{v}_2$  are not multiples of each other, a basis for the image of  $T_B$  is  $\{\bar{v}_1, \bar{v}_2\}$ . Therefore, a basis for the image of  $T$  is  $\{K_B^{-1}(\bar{v}_1), K_B^{-1}(\bar{v}_2)\} = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}$ .