MAS 3105 (Linear Algebra) Your name: KEY

Quiz 6

November 5, 2013

1. Let V be a vector space and let $\mathcal{B}, \mathcal{C}$ be two bases for V. Define the change of basis matrix $S_{\mathcal{B}\to\mathcal{C}}$.

$$\begin{tikzcd} & \mathbb{R}^n \\ V \arrow[ur, "K_{\mathcal{C}}"] \arrow[dr, "K_{\mathcal{B}}"'] & \\ & \mathbb{R}^n \arrow[uu] \end{tikzcd}$$

Change of basis matrix $S_{\mathcal{B}\to\mathcal{C}}$ is the standard matrix of a linear transformation $K_{\mathcal{C}} \circ K_{\mathcal{B}}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, where $K_{\mathcal{B}}$ and $K_{\mathcal{C}}$ are coordinate isomorphisms.

2. Let $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation defined by

$$T(M) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} M$$

(a) Find the matrix $T_{\mathcal{B}}$ of T with respect to a basis

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{b_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{b_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{b_4} \right\}.$$

$$\left.\begin{aligned} T(b_1) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \\ T(b_2) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \\ T(b_3) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \\ T(b_4) &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \end{aligned}\right\} \Rightarrow T_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

(b) Determine whether T is an isomorphism. Explain your answer. If T is not an isomorphism, find a basis for the image of T.

Since $\operatorname{rank} T_{\mathcal{B}} = 2$, T is not an isomorphism. If $\bar{v}_i$ denotes the i-th column of $T_{\mathcal{B}}$, then $\bar{v}_3 = \bar{v}_1$ and $\bar{v}_4 = \bar{v}_2$. Since $\bar{v}_1$ and $\bar{v}_2$ are not multiples of each other, a basis for the image of $T_{\mathcal{B}}$ is $\{\bar{v}_1, \bar{v}_2\}$. Therefore, a basis for the image of T is $\left\{K_{\mathcal{B}}^{-1}(\bar{v}_1), K_{\mathcal{B}}^{-1}(\bar{v}_2)\right\} = \left\{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right\}$.