Supplementary handout on the one-to-one, onto maps and one-to-one correspondences

MAS 3105, Dmytro Savchuk

September 10, 2013

This handout contains the definitions and statements required for the course that are not covered by the Otto Bretscher's book.

Definition 1 (informal)**.** *A* function *from the set X to the set Y is a rule assigning exactly one element of Y to each element of X.*

For a function from X to Y we use a notation

$$f\colon X \to Y.$$

Definition 2. *Let $f\colon X \to Y$ be a function.*

- *The set X is called the* domain *of f;*
- *The set Y is called the* codomain *of f.*
- *The set $f(X) = \{f(x)\colon x \in X\}$ is called the* image *(or the* range*) of f.*

The last definition is illustrated in Figure 1.

Definition 3. *A* function *from the set X to the set Y is called*

- one-to-one*, if distinct elements of X are mapped to distinct elements of Y. I.e. if $x_1 \neq x_2$ are different elements of X, then $f(x_1) \neq f(x_2)$;*
- onto*, if for each $y \in Y$ there is $x \in X$ such that $f(x) = y$;*
- one-to-one correspondence *if it is both one-to-one and onto.*

The examples illustrating the above notions are shown in Figure 2.

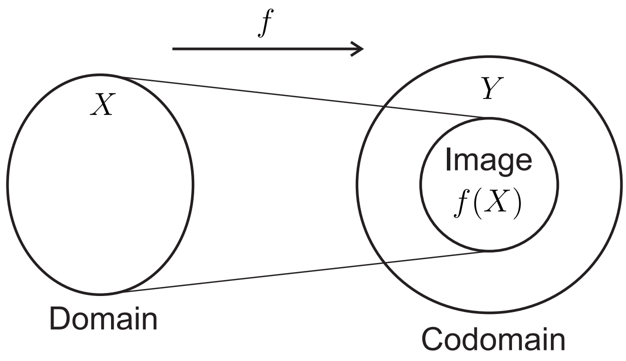

Figure 1: Relations between domain, codomain and the image

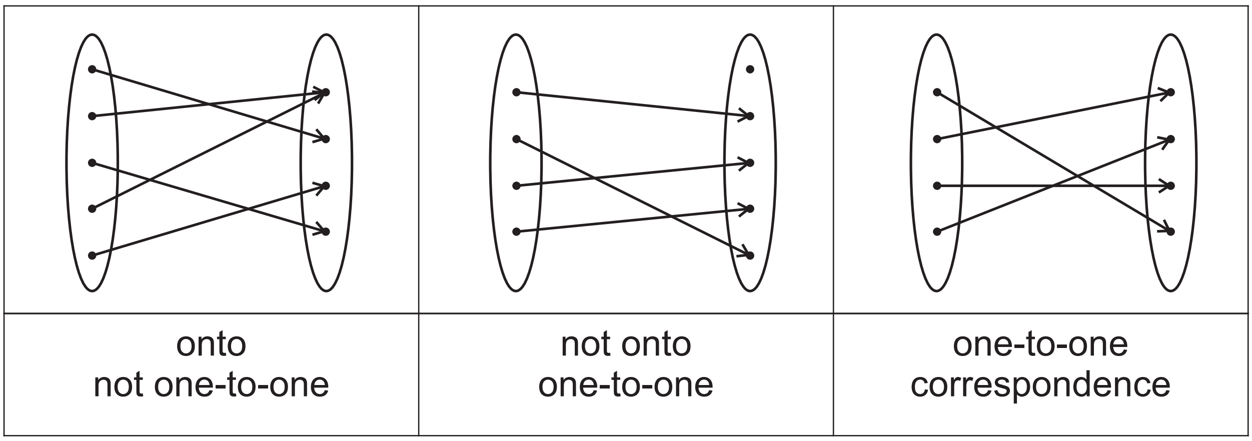

Figure 2: Examples of functions

Example 1.

- *$f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither one-to-one ($f(-1) = f(1) = 1$) nor onto (there is no $x \in \mathbb{R}$ such that $f(x) = -1$)*
- *$f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is one-to-one, but not onto (there is no $x \in [0,\infty)$ such that $f(x) = -1$)*
- *$f\colon \mathbb{R} \to [0,\infty)$ defined by $f(x) = x^2$ is onto, but not one-to-one ($f(-1) = f(1) = 1$)*
- *$f\colon [0,\infty) \to [0,\infty)$ defined by $f(x) = x^2$ is both one-to-one and onto, so is a one-to-one correspondence*

Definition 4. *Suppose $f\colon X \to Y$ is a one-to-one correspondence. The function $f^{-1}\colon Y \to X$, sending each $y \in Y$ to the unique $x \in X$ such that $f(x) = y$, is called the* inverse function *to the function f.*

Proposition 2. *Let $f\colon X \to Y$ be a one-to-one correspondence, and let $f^{-1}\colon Y \to X$ be its inverse function. Then*

(i) $f^{-1}(f(x)) = x$ *for each* $x \in X$*;*

(ii) $f(f^{-1}(y)) = y$ *for each* $y \in Y$*;*

(iii) f *is the inverse function to* f^{-1}*.*

Proposition 3. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined by an $m \times n$ matrix A. Then*

(i) T *is one-to-one* $\Leftrightarrow rank(A) = n$ *(number of columns of A);*

(ii) T *is onto* $\Leftrightarrow rank(A) = m$ *(number of rows of A);*

(iii) T *is one-to-one correspondence* $\Leftrightarrow m = rank(A) = n$ *(in particular, A is a square matrix).*

Proof. (i) T is one-to-one $\Leftrightarrow T(\bar{x}) = \bar{b}$ has at most one solution for each $\bar{b} \in \mathbb{R}^m \Leftrightarrow A\bar{x} = \bar{b}$ has at most one solution for each $\bar{b} \in \mathbb{R}^m \Leftrightarrow rankA = n$ (the system does not have free variables)

(i) T is onto $\Leftrightarrow T(\bar{x}) = \bar{b}$ has a solution for each $\bar{b} \in \mathbb{R}^m \Leftrightarrow A\bar{x} = \bar{b}$ has a solution for each $\bar{b} \in \mathbb{R}^m \Leftrightarrow rankA = m$ (last column of $ref(A)$ cannot be pivot)

(iii) follows from (i) and (ii). □