Solutions to practice problems for the Final

Systems of linear equations

1. Find the general solution of the following system of equations.

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5 \\ + \ \ x_2 + x_3 + 2x_4 + x_5 &= 4 \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3 \end{aligned}$$

SOLUTION.

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & -2 & -2 & -3 & -1 & -7 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & -6 \\ 0 & 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$x_1 = -6 + 2s + t, \quad x_2 = 2 - s + t, \quad x_3 = s, \quad x_4 = 1 - t, \quad x_5 = t$$

2. Solve the following system of linear equations.

$$\begin{aligned} 2x_1 + \ \ x_2 + 3x_3 &= \ \ 1 \\ 4x_1 + 3x_2 + 5x_3 &= \ \ 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned}$$

SOLUTION.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -4 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$x_1 = -3, \quad x_2 = 1, \quad x_3 = 2$$

3. Consider the following system of linear equations.

$$\begin{aligned} x_1 + x_2 + 3x_3 &= a \\ 2x_1 + x_2 + 4x_3 &= b \\ 3x_1 + x_2 + 5x_3 &= c \end{aligned}$$

For any fixed values of a, b, and c we obtain a system of 3 equations in 3 unkowns.

(a) Find a set of values for a, b, and c so that the system is inconsistent.

(b) For $a = 0$, $b = 1$, $c = 2$, find the general form of the solution.

Solution. Consider the following row reduction

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 2 & 1 & 4 & b \\ 3 & 1 & 5 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & -1 & -2 & -2a+b \\ 0 & -2 & -4 & -3a+c \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & -1 & -2 & -2a+b \\ 0 & 0 & 0 & a-2b+c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 1 & 2 & 2a-b \\ 0 & 0 & 0 & a-2b+c \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & -a+b \\ 0 & 1 & 2 & 2a-b \\ 0 & 0 & 0 & a-2b+c \end{array}\right]$$

(a) From the last line of the last matrix above we see that any choice of a, b, c with $a - 2b + c \neq 0$ makes the system inconsistent. For a specific example we can take $a = 1$, $b = c = 0$.

(b) For $a = 0$, $b = 1$, $c = 2$ the system is consistent and the general form of the solution of the system is $x_1 = 1 - t$, $x_2 = -1 - 2t$, $x_3 = t$.

4. Explain why a homogeneous system of linear equations with more unkowns than equations always has nontrivial solutions.

Solution.

Matrix algebra

5. Either compute the inverse of the matrix A below, or explain why A is not invertible.

$$A = \begin{bmatrix} 2 & 5 & 8 & 5 \\ 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 2 \\ 1 & 3 & 5 & 3 \end{bmatrix}$$

Solution. We row reduce the matrix $\left[\, A \mid I \,\right]$.

$$\left[\begin{array}{cccc|cccc} 2 & 5 & 8 & 5 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 7 & 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 0 & 1 & 0 & 0 \\ 2 & 5 & 8 & 5 & 1 & 0 & 0 & 0 \\ 2 & 4 & 7 & 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & -1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & -1 & 8 & -3 & 1 \\ 0 & 1 & 0 & 0 & -2 & 5 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -2 & 1 & -5 \\ 0 & 1 & 0 & 0 & -2 & 5 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 3 & -2 & 1 & -5 \\ -2 & 5 & -2 & 3 \\ 0 & -2 & 1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}$

6. Either compute the inverse of the matrix B below, or explain why B is not invertible.

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 3 \\ -2 & 0 & 2 \end{bmatrix}$$

Solution. We row reduce the matrix $\left[\begin{array}{c|c} B & I \end{array}\right]$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ -2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & 2 & 6 & 2 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \end{array}\right]$$

This shows that the rank of B is 2, so B is not invertible.

7. Find all values of λ for which the matrix below is singular (i.e., not invertible).

$$\begin{bmatrix} \lambda & 1 & -1 \\ 1 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix}$$

Solution. The determinant of the matrix is

$$\det \begin{bmatrix} \lambda & 1 & -1 \\ 1 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} = \lambda(0-(-2)) - (0-2) + (-1)(1-(-2)) = 2\lambda - 1$$

The matrix is singular if and only if its determinant is 0. This happens if and only if $\lambda = 1/2$.

8. Find the determinant of the matrix

$$Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Solution.

$$\det Q = \det \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = (1)(-2)(2) = -4$$

9. Find the determinant of the matrix

$$N = \begin{bmatrix} 3 & 4 & 5 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 3 & 6 & 3 \\ 7 & 2 & 9 & 4 \end{bmatrix}$$

SOLUTION.

$$\det N = \det \begin{bmatrix} 3 & 4 & 5 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 3 & 6 & 3 \\ 7 & 2 & 9 & 4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 6 & 3 \\ 7 & 2 & 9 & 4 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 4 & 2 & 2 \\ 0 & 3 & 4 & 3 \\ 0 & 2 & 2 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 3 & 4 & 3 \\ 0 & 4 & 2 & 2 \end{bmatrix}$$

$$= 2\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 4 & 3 \\ 0 & 4 & 2 & 2 \end{bmatrix} = 2\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

$$= 2\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -12 \end{bmatrix} = 2(-12) = -24$$

10. Let A be a 4×4 matrix with rows $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_4$, in that order. If $\det(A) = 4$, find

$$\det \begin{bmatrix} \vec{r}_2 - 2\vec{r}_1 \\ \vec{r}_1 \\ \vec{r}_3 \\ 3\vec{r}_4 \end{bmatrix}.$$

SOLUTION.

$$\det \begin{bmatrix} \vec{r}_2 - 2\vec{r}_1 \\ \vec{r}_1 \\ \vec{r}_3 \\ 3\vec{r}_4 \end{bmatrix} = -\det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 - 2\vec{r}_1 \\ \vec{r}_3 \\ 3\vec{r}_4 \end{bmatrix} = -\det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ 3\vec{r}_4 \end{bmatrix} = -3\det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vec{r}_4 \end{bmatrix} = -3(4) = -12$$

11. If A and B are square matrices of the same size and $\det A = 2$ and $\det B = 3$, find $\det(A^2B^{-1})$.

SOLUTION.

$$\det(A^2B^{-1}) = \big(\det(A)\big)^2 \frac{1}{\det(B)} = 2^2\frac{1}{3} = \frac{4}{3}$$

Vector spaces

12. Let $X = (\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1 = (1,2,0,4)$ and $\vec{v}_2 = (1,1,1,3)$. Determine which of the two vectors $\vec{w}_1 = (1,4,-2,6)$ and $\vec{w}_2 = (2,6,0,9)$ is in $\mathrm{Span}(X)$ and write that vector as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

    SOLUTION. We row reduce the matrix $\left[\ \vec{v}_1\ \vec{v}_2 \mid \vec{w}_1 \mid \vec{w}_2\ \right]$.

$$\left[\begin{array}{cc|c|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 6 \\ 0 & 1 & -2 & 0 \\ 4 & 3 & 6 & 9 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

    This shows that $\vec{w}_1$ is in $\mathrm{Span}(X)$ and $\vec{w}_2$ is not in $\mathrm{Span}(X)$. Further, the unique way of writing $\vec{w}_1$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ is

$$\vec{w}_1 = 3\vec{v}_1 - 2\vec{v}_2.$$

13. Determine whether $q(x) = x^3 + x^2 - 3x + 2$ is a linear combination of $p_1(x) = x^3$, $p_2(x) = x^2 + 3x$, and $p_3(x) = x^2 + 1$ in P_3 and, if so, find scalars c_1, c_2, and c_3 such that $q(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x)$. Are these scalars unique?

    SOLUTION. We need to look for scalars c_1, c_2, c_3 so that

$$q(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x).$$

    Substituting the values of q, p_1, p_2, p_3 we get

$$\begin{aligned} x^3 + x^2 - 3x + 2 &= c_1(x^3) + c_2(x^2 + 3x) + c_3(x^2 + 1) \\ &= (c_1)x^3 + (c_2 + c_3)x^2 + (3c_2)x + (c_3), \end{aligned}$$

    which is equivalent to the system of linear equations

$$\begin{array}{rcr} c_1 \qquad\quad & = & 1 \\ c_2 + c_3 & = & 1 \\ 3c_2 \qquad & = & -3 \\ c_3 & = & 2 \end{array}$$

    It is easy to see that the unique solution to this system is $c_1 = 1$, $c_2 = -1$, $c_3 = 2$. Thus,

$$q(x) = p_1(x) - p_2(x) + 2p_3(x).$$

14. Let $X = (\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$, where

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \vec{u}_4 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

    Show that the list X is linearly dependent by

    (a) finding a linear dependence relation on X.

    (b) writing one of the vectors in X as a linear combination of the other vectors in X.

SOLUTION. Let

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

The reduced row echelon form of A is

$$\mathcal{R}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since A has rank 3, the list X is linearly dependent. To find a specific linear dependence relation on X first note that the general solution of $A\vec{x} = \vec{0}$ is

$$x_1 = t,\ x_2 = -2t,\ x_3 = t,\ x_4 = t.$$

Using $t = 1$ we get the linear dependence relation

$$\vec{u}_1 - 2\vec{u}_2 + \vec{u}_3 + \vec{u}_4 = \vec{0}.$$

You can solve this relation for any one of the vectors.

$$\begin{aligned} \vec{u}_1 &= 2\vec{u}_2 - \vec{u}_3 - \vec{u}_4 \\ \vec{u}_2 &= \frac{1}{2}\vec{u}_1 + \frac{1}{2}\vec{u}_3 + \frac{1}{2}\vec{u}_4 \\ \vec{u}_3 &= -\vec{u}_1 + 2\vec{u}_2 - \vec{u}_4 \\ \vec{u}_4 &= -\vec{u}_1 + 2\vec{u}_2 - \vec{u}_3 \end{aligned}$$

15. Determine whether the list $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is linearly independent, where $\vec{v}_1 = (1, 3, 3)$, $\vec{v}_2 = (2, 2, 3)$, and $\vec{v}_3 = (3, 1, 3)$. If X is linearly dependent, then find a specific linear dependence relation on X.

SOLUTION. Let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$. We row reduce A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the rank of A is $2 < 3$, the list X is linearly dependent. From the reduced row echelon form of A we see that the general solution of the equation $A\vec{x} = \vec{0}$ is $x_1 = t$, $x_2 = -2t$, $x_3 = t$. Using $t = 1$ we get that

$$\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$$

is a specific linear dependence relation on X.

16. Determine whether the list $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is linearly independent, where $\vec{v}_1 = (1, 5, 2)$, $\vec{v}_2 = (1, 1, 7)$, and $\vec{v}_3 = (0, -3, 4)$. If X is linearly dependent, then find a specific linear dependence relation on X.

SOLUTION. Let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$. We row reduce A:

$$\begin{bmatrix} 1 & 1 & 0 \\ 5 & 1 & -3 \\ 2 & 7 & 4 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the rank of A is 3, A is invertible and hence the list X is linearly independent.

17. Let $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$, where $\vec{v}_1 = (2, 5, -3, 6)$, $\vec{v}_2 = (1, 0, 0, 1)$, and $\vec{v}_3 = (4, 0, 9, 6)$. Is X linearly independent? Does X span $\mathbb{R}^4$? Is X a basis for $\mathbb{R}^4$?

    SOLUTION. X is linearly independent. X does not span $\mathbb{R}^4$. X is not a basis for $\mathbb{R}^4$.

18. Let $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$, where $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (2, 0, 1)$, and $\vec{v}_3 = (3, 1, 0)$. Is X linearly independent? Does X span $\mathbb{R}^3$? Is X a basis for $\mathbb{R}^3$?

    SOLUTION. X is linearly independent. X does span $\mathbb{R}^3$. X is a basis for $\mathbb{R}^3$.

19. Let $X = (p_1(x), p_2(x), p_3(x))$, where $p_1(x) = x^2 + 1$, $p_2(x) = x + 1$, and $p_3(x) = x^2 + x$. Is X linearly independent? Does X span P_3? Is X a basis for P_3?

    SOLUTION. Let X is linearly independent in P_3. X does not span P_3. X is not a basis for P_3. (Note that X does span P_2 and hence is a basis or P_2.)

20. Consider the list $S = \big((3, 0, 0, -1), (3, 3, 3, 2), (0, 1, 1, 1), (0, 1, 2, 1)\big)$ of vectors in $\mathbb{R}^4$. Let $W = \text{Span}(S)$ be the subspace of $\mathbb{R}^4$ spanned by S. Find a basis for W.

    SOLUTION. Let $A = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ -1 & 2 & 1 & 1 \end{bmatrix}$. Then $\text{Span}(S) = \text{Col}\, A$. The reduced row echelon form of A is

$$\mathcal{R}(A) = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

    Since the pivot columns of A are a basis for $\text{Col}\, A$ and this shows that the pivot columns of A are the first, second, and fourth columns, a basis for the span of S is $X = \big((3, 0, 0, -1), (3, 3, 3, 2), (0, 1, 2, 1)\big)$.

    Alternatively, let $B = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 3 & 3 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$. Then $\text{Span}(S) = \text{Row}\, A$. The reduced row echelon form of B is

$$\mathcal{R}(B) = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

    Since the nonzero rows of $\mathcal{R}(B)$ are a basis for $\text{Row}\, B$, a basis for the span of S is $Y = \big((1, 0, 0, -1/3), (0, 1, 0, 1), (0, 0, 1, 0)\big)$.

21. Let $A = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{bmatrix}$.

(a) Find a basis, X, for the column space of A.

(b) Find a basis, Y, for the null space of A.

(c) Find a basis, Z, for the row space of A.

SOLUTION. The reduced row echelon form of A is

$$\mathcal{R}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) $X = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \end{bmatrix} \right)$

(b) $Y = \left(\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right)$

(c) $Z = \left(\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -5 \end{bmatrix} \right)$

22. Let $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$.

(a) Find a basis for the column space of A.

(b) Find a basis for the null space of A.

(c) Find a basis for the row space of A.

SOLUTION. (a) A basis for the column space of A is $X = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right)$

(b) A basis for the null space of A is $Y = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$

(c) A basis for the row space of A is $Z = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$

23. If the null space of a 5×6 matrix A is 4-dimensional,

(a) what is the dimension of the row space of A?

(b) for what value of k is the the column space of A a subspace of $\mathbb{R}^k$?

(c) for what value of m is the the null space of A a subspace of $\mathbb{R}^m$?

SOLUTION. If the null space of a 5×6 matrix A is 4-dimensional,

(a) the dimension of the row space of A is 2,

(b) the column space of A is a subspace of $\mathbb{R}^5$, and

(c) the null space of A is a subspace of $\mathbb{R}^6$.

24. A 26×37 matrix has a 13 dimensional nullspace. What is the rank of the matrix?

SOLUTION. If A is a 26×37 matrix with a 13 dimensional nullspace, then the rank of A is $37 - 13 = 24$.

25. Consider a linear system $A\vec{x} = \vec{b}$. If A is 7×4 and the dimension of the null space of A is 0, how many solutions can this system have?

SOLUTION. If A is a 7×4 whose nullity (the dimension of the null space) is 0, then any linear system $A\vec{x} = \vec{b}$ can have either 0 or 1 solution.

26. Consider the bases $X = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and $Y = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ of $\mathbb{R}^3$, where $\vec{u}_1 = (2, 1, 1)$, $\vec{u}_2 = (2, -1, 1)$, $\vec{u}_3 = (1, 2, 1)$, $\vec{v}_1 = (3, 1, -5)$, $\vec{v}_2 = (1, 1, -3)$, and $\vec{v}_3 = (-1, 0, 2)$.

(a) Find the change of basis matrix, ${}_Y I_X$, that changes from coordinates with respect to X to coordinates with respect to Y.

(b) Find the change of basis matrix, ${}_X I_Y$, that changes from coordinates with respect to Y to coordinates with respect to X.

(c) Compute the coordinate vector, $K_Y(\vec{w})$, of $\vec{w} = (-5, 8, -5)$ with respect to the basis Y.

(d) Use your answers from parts (b) and (c) to compute the coordinate vector, $K_X(\vec{w})$, of $\vec{w}$ with respect to the basis X.

(e) Check your work by directly computing the coordinate vector of $\vec{w}$ with respect to the basis X.

SOLUTION. (a) By definition,

$$ {}_Y I_X = \begin{bmatrix} K_Y(\vec{u}_1) & K_Y(\vec{u}_2) & K_Y(\vec{u}_3) \end{bmatrix} = \begin{bmatrix} K_Y\big((2,1,1)\big) & K_Y\big((2,-1,1)\big) & K_Y\big((1,2,1)\big) \end{bmatrix} $$

Now, $K_Y\big((x_1, x_2, x_3)\big) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ if and only if

$$ y_1(3, 1, -5) + y_2(1, 1, -3) + y_3(-1, 0, 2) = (x_1, x_2, x_3) $$

This is equivalent to the system of linear equations

$$\begin{aligned} 3y_1 + y_2 - y_3 &= x_1 \\ y_1 + y_2 &= x_2 \\ -5y_1 - 3y_2 + 2y_3 &= x_3 \end{aligned}$$

We can calculate $K_Y(\vec{u}_1)$, $K_Y(\vec{u}_2)$, and $K_Y(\vec{u}_3)$ simultaneously by row reducing the triply augmented matrix $\left[\ \vec{v}_1\ \vec{v}_2\ \vec{v}_3 \mid \vec{u}_1 \mid \vec{u}_2 \mid \vec{u}_3\ \right]$.

$$\left[\begin{array}{ccc|c|c|c} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c|c|c} 1 & 1 & 0 & 1 & -1 & 2 \\ 3 & 1 & -1 & 2 & 2 & 1 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c|c|c} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & -2 & -1 & -1 & 5 & -5 \\ 0 & 2 & 2 & 6 & -4 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c|c|c} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & -2 & -1 & -1 & 5 & -5 \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c|c|c} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c|c|c} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

So

$${}_Y I_X = \left[\ K_Y(\vec{u}_1) \quad K_Y(\vec{u}_2) \quad K_Y(\vec{u}_3)\ \right] = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$$

(b) $${}_X I_Y = \begin{bmatrix} \frac{35}{2} & \frac{19}{2} & -\frac{13}{2} \\ -\frac{19}{2} & -\frac{11}{2} & \frac{7}{2} \\ -13 & -7 & 5 \end{bmatrix}$$

(c) By definition, $K_Y(\vec{w}) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ if and only if $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}$.

Substituting the values of the vectors in this equation gives

$$c_1(3, 1, -5) + y_2(1, 1, -3) + y_3(-1, 0, 2) = (-5, 8, -5).$$

This is equivalent to the system of linear equations

$$\begin{aligned} 3y_1 + y_2 - y_3 &= -5 \\ y_1 + y_2 &= 8 \\ -5y_1 - 3y_2 + 2y_3 &= -5 \end{aligned}$$

We can calculate $K_Y(\vec{w})$ by row reducing the augmented matrix $\left[\ \vec{v}_1\ \vec{v}_2\ \vec{v}_3 \mid \vec{w}\right]$.

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & -5 \\ 1 & 1 & 0 & 8 \\ -5 & -3 & 2 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 3 & 1 & -1 & -5 \\ -5 & -3 & 2 & -5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & -2 & -1 & -29 \\ 0 & 2 & 2 & 35 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & -2 & -1 & -29 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & 1 & \frac{1}{2} & \frac{29}{2} \\ 0 & 0 & 1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & \frac{23}{2} \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & \frac{23}{2} \\ 0 & 0 & 1 & 6 \end{array}\right]$$

So

$$K_Y(\vec{w}) = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$$

(d) $K_X(\vec{w}) = ({}_X I_Y) K_Y(\vec{w}) = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$

(e) By definition, $K_X(\vec{w}) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ if and only if $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{w}$.

Substituting the values of the vectors in this equation gives

$$c_1(2,1,1) + y_2(2,-1,1) + y_3(1,2,1) = (-5,8,-5).$$

This is equivalent to the system of linear equations

$$\begin{aligned} 2y_1 + 2y_2 + y_3 &= -5 \\ y_1 - y_2 + 2y_3 &= 8 \\ y_1 + y_2 + y_3 &= -5 \end{aligned}$$

We can calculate $K_X(\vec{w})$ by row reducing the augmented matrix $\left[\begin{array}{ccc|c} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{w} \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & -5 \\ 1 & -1 & 2 & 8 \\ 1 & 1 & 1 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 2 & 2 & 1 & -5 \\ 1 & 1 & 1 & -5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & 4 & -3 & -21 \\ 0 & 2 & -1 & -13 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & 2 & -1 & -13 \\ 0 & 4 & -3 & -21 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & 2 & -1 & -13 \\ 0 & 0 & -1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & 1 & -\frac{1}{2} & -\frac{13}{2} \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 18 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

So

$$K_X(\vec{w}) = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

27. Let $X = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$, where

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

and let $E = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the standard basis for $\mathbb{R}^3$.

(a) Show that X is a basis for $\mathbb{R}^3$.

(b) Find the change of basis matrix, ${}_E I_X$, that changes from coordinates with respect to X to coordinates with respect to E.

(c) Find the change of basis matrix, ${}_X I_E$, that changes from coordinates with respect to E to coordinates with respect to X.

SOLUTION. We know that ${}_X I_E = ({}_E I_X)^{-1}$ and that ${}_E I_X = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$. We compute this inverse in the usual way

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

So

$${}_X I_E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Linear transformations

28. If $T\colon \mathbb{R}^5 \to \mathbb{R}^2$ is a linear transformation, what are the possible dimensions for the kernel of T? For each of these possibilities, what is the dimension of the image of T?

SOLUTION. If $T\colon \mathbb{R}^5 \to \mathbb{R}^2$ is a linear transformation, then either $\dim \operatorname{Kernel} T = 3$, $\dim \operatorname{Image} T = 2$ or $\dim \operatorname{Kernel} T = 4$, $\dim \operatorname{Image} T = 1$ or $\dim \operatorname{Kernel} T = 5$, $\dim \operatorname{Image} T = 0$.

29. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 - 3x_2 \\ 3x_1 - x_2 \end{bmatrix}$$

(a) Find the matrix of F with respect to the standard bases on $\mathbb{R}^2$ and $\mathbb{R}^3$.

(b) Determine whether F is one-to-one and, if not, find a basis the kernel of F.

(c) Determine whether F is onto and, if not, find a basis for the image of F.

(d) Find the matrix of F with respect to the bases $X = (\vec{v}_1, \vec{v}_2)$ for $\mathbb{R}^2$ and $Y = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$ for $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \vec{w}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

SOLUTION.

(a) Let E_3 and E_2 be the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Then the matrix of F with respect to E_3 and E_2 is

$$_{E_3}F_{E_2} = \left[\ K_{E_3}\big(F(\vec{e}_1)\big)\ K_{E_3}\big(F(\vec{e}_2)\big)\ \right] = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 3 & -1 \end{bmatrix}$$

(b) The rank of the matrix $_{E_3}F_{E_2}$ is 2, so F is one-to-one but not onto.

30. Consider the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$, given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ -3x_1 + 6x_2 + x_3 \\ -x_1 + 2x_2 \\ 2x_1 - 4x_2 + x_3 \end{bmatrix}$$

(a) Find the matrix of T with respect to the standard bases on $\mathbb{R}^3$ and $\mathbb{R}^4$.

(b) Determine whether T is one-to-one and, if not, find a basis the kernel of T.

(c) Determine whether T is onto and, if not, find a basis for the image of T.

(d) Find the matrix of T with respect to the bases $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ for $\mathbb{R}^3$ and $Y = (\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4)$ for $\mathbb{R}^4$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

and

$$\vec{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \ \vec{w}_3 = \begin{bmatrix} -5 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \ \vec{w}_4 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}.$$

SOLUTION. (a) Let E_3 and E_4 be the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively. Then the matrix of F with respect to E_3 and E_4 is

$$_{E_4}F_{E_2} = \left[\ K_{E_4}\big(F(\vec{e}_1)\big)\ K_{E_4}\big(F(\vec{e}_2)\big)\ K_{E_4}\big(F(\vec{e}_3)\big)\ \right] = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 6 & 1 \\ -1 & 2 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

(b) The reduced row echelon form of ${}_{E_4}F_{E_2}$ is

$$\mathcal{R}({}_{E_4}F_{E_2}) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So a basis for the column space of ${}_{E_4}F_{E_2}$ is

$$X = \left(\begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

and a basis for the null space of ${}_{E_4}F_{E_2}$ is

$$Y = \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right)$$

Since the coordinate transformations K_{E_3} and K_{E_4} are the identity transformations on $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively, X and Y are also bases for the image and kernel of F, respectively.

31. Let $X = (p_1(x), p_2(x), p_3(x))$, where

$$p_1(x) = 1 + 2x, \quad p_2(x) = x - x^2, \quad p_3(x) = x + x^2,$$

and let $T\colon P_2 \to P_2$ be the linear transformation defined by $T(p(x)) = \frac{d}{dx}p(x) - p(x)$.

(a) Show that X is a basis for P_2.

(b) Let $q(x) = 1 + 3x + x^2$. Compute $K_X(q(x))$.

(c) Find the matrix of T with respect to the basis X.

(d) Find the matrix of T with respect to the standard basis, $S = (1, x, x^2)$, of P_2.

(e) Verify that ${}_XT_X = ({}_XI_E)({}_ET_E)({}_EI_X)$.

(f) Use your answers from parts (b) and (c) to find $K_X(T(q(x)))$.

(g) Compute $K_E(T(q(x)))$.

(h) Verify that $K_X(T(q(x))) = {}_XI_E \cdot K_E(T(q(x)))$.

Solution. (a) Rank $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = 3$

(b) $K_X(q(x)) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(c) ${}_XT_X = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

(d) ${}_ET_E = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

(f) $K_X(T(q(x))) = {}_XT_XK_X(T(q(x))) = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$

(g) $K_E(T(q(x))) = K_E((3+2x) - (1+3x+x^2)) = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

(h) ${}_XI_EK_E(T(q(x))) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$

Eigenvalues and eigenvectors

For each of the matrices below

(a) calculate the characteristic polynomial of A,

(b) find the eigenvalues of A,

(c) find a basis for each eigenspace of A, and

(d) determine whether or not A is diagonalizable. If A is diagonalizable, then find a matrix P such that $P^{-1}AP$ is diagonal. If not, explain why A is not diagonalizable.

32. $A = \begin{bmatrix} -12 & -5 \\ 29 & 12 \end{bmatrix}$

33. $A = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$

34. $A = \begin{bmatrix} 11 & 25 \\ -4 & -9 \end{bmatrix}$

35. $A = \begin{bmatrix} -1 & -3 & -3 \\ 3 & 5 & 3 \\ -1 & -1 & 1 \end{bmatrix}$

36. $A = \begin{bmatrix} 2 & 5 & 10 \\ 1 & 2 & 4 \\ -1 & -1 & -4 \end{bmatrix}$

37. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 5 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

38. $A = \begin{bmatrix} 5 & 4 & -3 \\ -4 & -3 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

39. $A = \begin{bmatrix} 2 & -2 & 5 \\ -3 & 1 & -5 \\ -3 & 2 & -6 \end{bmatrix}$

Solution.

1. (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{bmatrix} -12-\lambda & -5 \\ 29 & 12-\lambda \end{bmatrix} \\ &= (-12-\lambda)(12-\lambda)-(-5)(29) \\ &= \lambda^2+1\end{aligned}$$

   (b) A has no eigenvalues.

   (c) A has no eigenvectors because it has no eigenvalues.

   (d) A is not diagonalizable since the sum of the dimensions of its eigenspaces is $0 < 2$.

2. (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{bmatrix} -1-\lambda & -2 \\ 6 & 6-\lambda \end{bmatrix} \\ &= (-1-\lambda)(6-\lambda)-(-2)(6) \\ &= \lambda^2-5\lambda+6 \\ &= (\lambda-2)(\lambda-3)\end{aligned}$$

   (b) A has eigenvalues 2 and 3, each with multiplicity 1.

   (c) The eigenspace of A associated to the eigenvalue 2 is the null space of the matrix $A-2I$. To find a basis for this eigenspace we row reduce this matrix.

$$A-2I = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

   So the general solution to the equation $(A-2I)\vec{x}=\vec{0}$ is $x_1 = -\frac{2}{3}t$, $x_2 = t$. Using $t=3$ we get that $X_2 = \left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$ is a basis for the eigenspace of A associated to the eigenvalue 2.

   The eigenspace of A associated to the eigenvalue 3 is the null space of the matrix $A-3I$. To find a basis for this eigenspace we row reduce this matrix.

$$A-3I = \begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

   So the general solution to the equation $(A-3I)\vec{x}=\vec{0}$ is $x_1 = -\frac{1}{2}t$, $x_2 = t$. Using $t=2$ we get that $X_3 = \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$ is a basis for the eigenspace of A associated to the eigenvalue 3.

   (d) A is diagonalizable since the sum of the dimensions of its eigenspaces of A is $1+1=2$. Further, if we set

$$P = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix},$$

   then P is invertible and

$$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

3. (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{bmatrix} 11-\lambda & 25 \\ -4 & -9-\lambda \end{bmatrix} \\ &= (11-\lambda)(-9-\lambda)-(25)(-4) \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda-1)^2\end{aligned}$$

(b) A has eigenvalue 1, with multiplicity 2.

(c) The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A-I$. To find a basis for this eigenspace we row reduce this matrix.

$$A - I = \begin{bmatrix} 10 & 25 \\ -4 & -10 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A-I)\vec{x} = \vec{0}$ is $x_1 = -\frac{5}{2}t$, $x_2 = t$. Using $t = 2$ we get that $X_1 = \left(\begin{bmatrix} -5 \\ 2 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue 1.

(d) A is not diagonalizable since the sum of the dimensions of its eigenspaces is $1 < 2$.

4. (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \det \begin{bmatrix} -1-\lambda & -3 & -3 \\ 3 & 5-\lambda & 3 \\ -1 & -1 & 1-\lambda \end{bmatrix} \\ &= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \\ &= -(\lambda-1)(\lambda-2)^2\end{aligned}$$

(b) A has eigenvalues 1 and 2, with multiplicities 1 and 2, respectively.

(c) The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A-I$. To find a basis for this eigenspace we row reduce this matrix.

$$A - I = \begin{bmatrix} -2 & -3 & -3 \\ 3 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A-I)\vec{x} = \vec{0}$ is $x_1 = 3t$, $x_2 = -3t$, $x_3 = t$. Using $t = 1$ we get that $X_1 = \left(\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue 1.

The eigenspace of A associated to the eigenvalue 2 is the null space of the matrix $A-2I$. To find a basis for this eigenspace we row reduce this matrix.

$$A - 2I = \begin{bmatrix} -3 & -3 & -3 \\ 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A-2I)\vec{x}=\vec{0}$ is $x_1=-s-t$, $x_2=s$, $x_3=t$. Using $s=1$, $t=0$ and then $s=0$, $t=1$ we get that $X_2=\left(\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$ is a basis for the eigenspace of A associated to the eigenvalue 2.

(d) A is diagonalizable since the sum of the dimensions of its eigenspaces is $1+2=3$. Further, if we set

$$P=\begin{bmatrix}3&-1&-1\\-3&1&0\\1&0&1\end{bmatrix},$$

then P is invertible and

$$P^{-1}AP=\begin{bmatrix}1&0&0\\0&2&0\\0&0&2\end{bmatrix}$$

5. (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I)&=\det\begin{bmatrix}2-\lambda&5&10\\1&2-\lambda&4\\-1&-1&-4-\lambda\end{bmatrix}\\&=-\lambda^3+3\lambda+2\\&=-(\lambda+1)^2(\lambda-2)\end{aligned}$$

(b) A has eigenvalues -1 and 2, with multiplicities 2 and 1, respectively.

(c) The eigenspace of A associated to the eigenvalue -1 is the null space of the matrix $A-(-1)I$. To find a basis for this eigenspace we row reduce this matrix.

$$A-(-1)I=\begin{bmatrix}3&5&10\\1&3&4\\-1&-1&-3\end{bmatrix}\to\cdots\to\begin{bmatrix}1&0&\frac{5}{2}\\0&1&\frac{1}{2}\\0&0&0\end{bmatrix}$$

So the general solution to the equation $(A-(-1)I)\vec{x}=\vec{0}$ is $x_1=-\frac{5}{2}t$, $x_2=-\frac{1}{2}t$, $x_3=t$. Using $t=2$ we get that $X_{-1}=\left(\begin{bmatrix}-5\\-1\\2\end{bmatrix}\right)$ is a basis for the eigenspace of A associated to the eigenvalue -1.

The eigenspace of A associated to the eigenvalue 2 is the null space of the matrix $A-2I$. To find a basis for this eigenspace we row reduce this matrix.

$$A-2I=\begin{bmatrix}0&5&10\\1&0&4\\-1&-1&-6\end{bmatrix}\to\cdots\to\begin{bmatrix}1&0&4\\0&1&2\\0&0&0\end{bmatrix}$$

So the general solution to the equation $(A - 2I)\vec{x} = \vec{0}$ is $x_1 = -4t$, $x_2 = -2t$, $x_3 = t$. Using $t = 1$ we get that $X_2 = \left(\begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue 2.

(d) A is not diagonalizable since the sum of the dimensions of its eigenspaces is $1 + 1 < 3$.

6. (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 5-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{bmatrix} \\ &= - - \lambda^3 + 5\lambda^2 + \lambda - 5 \\ &= -(\lambda + 1)(\lambda - 1)(\lambda - 5) \end{aligned}$$

(b) A has eigenvalues -1, 1, and 5, each with multiplicity 1.

(c) The eigenspace of A associated to the eigenvalue -1 is the null space of the matrix $A - (-1)I$. To find a basis for this eigenspace we row reduce this matrix.

$$A - (-1)I = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 6 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{12} \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A - (-1)I)\vec{x} = \vec{0}$ is $x_1 = \frac{1}{2}t$, $x_2 = -\frac{11}{12}t$, $x_3 = t$. Using $t = 12$ we get that $X_{-1} = \left(\begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue -1.

The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A - I$. To find a basis for this eigenspace we row reduce this matrix.

$$A - I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 4 & 5 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A - I)\vec{x} = \vec{0}$ is $x_1 = -4t$, $x_2 = t$, $x_3 = 0$. Using $t = 1$ we get that $X_1 = \left(\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue 1.

The eigenspace of A associated to the eigenvalue 5 is the null space of the matrix $A - 5I$. To find a basis for this eigenspace we row reduce this matrix.

$$A - 5I = \begin{bmatrix} -4 & 0 & -1 \\ 1 & 0 & 5 \\ 0 & 0 & -6 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A - 5I)\vec{x} = \vec{0}$ is $x_1 = 0$, $x_2 = t$, $x_3 = 0$. Using $t = 1$ we get that $X_5 = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue 5.

(d) A is diagonalizable since the sum of the dimensions of its eigenspaces is $1 + 1 + 1 = 3$. Further, if we set

$$P = \begin{bmatrix} 6 & -4 & 0 \\ -11 & 1 & 1 \\ 12 & 0 & 0 \end{bmatrix},$$

then P is invertible and

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

7. (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \begin{bmatrix} 5-\lambda & 4 & -3 \\ -4 & -3-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{bmatrix} \\ &= -\lambda^3 + \lambda^2 - \lambda + 1 \\ &= -(\lambda - 1)(\lambda^2 + 1) \end{aligned}$$

(b) The only eigenvalue of A is 1, with multiplicity 1.

(c) The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A - I$. To find a basis for this eigenspace we row reduce this matrix.

$$A - I = \begin{bmatrix} 4 & 4 & -3 \\ -4 & -4 & 2 \\ 2 & 2 & -2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A - I)\vec{x} = \vec{0}$ is $x_1 = -t$, $x_2 = t$, $x_3 = 0$. Using $t = 1$ we get that $X_1 = \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue 1.

(d) A is not diagonalizable since the sum of the dimensions of its eigenspaces is $1 < 3$.

8. (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & -2 & 5 \\ -3 & 1-\lambda & -5 \\ -3 & 2 & -6-\lambda \end{bmatrix} \\ &= -\lambda^3 - 3\lambda^2 - 3\lambda - 1 \\ &= -(\lambda + 1)^3 \end{aligned}$$

(b) The only eigenvalue of A is -1, with multiplicity 3.

(c) The eigenspace of A associated to the eigenvalue -1 is the null space of the matrix $A-(-1)I$. To find a basis for this eigenspace we row reduce this matrix.

$$A-(-1)I = \begin{bmatrix} 3 & -2 & 5 \\ -3 & 2 & -5 \\ -3 & 2 & -5 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution to the equation $(A-(-1)I)\vec{x} = \vec{0}$ is $x_1 = \frac{2}{3}s - \frac{5}{3}t$, $x_2 = s$, $x_3 = t$. Using $s = 3$, $t = 0$ and then $s = 0$, $t = 3$ we get that $X_{-1} = \left(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} \right)$ is a basis for the eigenspace of A associated to the eigenvalue -1.

(d) A is not diagonalizable since the sum of the dimensions of its eigenspaces is $2 < 3$.

Orthogonality

40. Let $X = (\vec{v}_1, \vec{v}_2)$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

and let $V = \text{Span}(X)$.

(a) Find a basis, $Y = (\vec{w}_1, \vec{w}_2)$, for the orthogonal complement of V in $\mathbb{R}^4$.

(b) Use the Gram-Schmidt process *on the list* X to produce an orthogonal basis, $X' = (\vec{v}_1', \vec{v}_2')$, for V.

(c) Use the Gram-Schmidt process *on the list* Y to produce an orthogonal basis, $Y' = (\vec{w}_1', \vec{w}_2')$, for $V^\perp$.

(d) Explain why $X' \cup Y' = (\vec{v}_1', \vec{v}_2', \vec{w}_1', \vec{w}_2')$ is an orthogonal basis for $\mathbb{R}^4$. (This doesn't require any further calculations.)

(e) Write the vector $\vec{u} = \begin{bmatrix} 2 \\ 7 \\ 1 \\ 3 \end{bmatrix}$ as a sum of two vectors, $\vec{u} = \vec{x} + \vec{y}$, where $\vec{x}$ is in V and $\vec{y}$ is in $V^\perp$. [*Hint.* Use $\vec{x} = \text{proj}_V \vec{u}$ and $\vec{y} = \vec{u} - \vec{x}$, or $\vec{y} = \text{proj}_{V^\perp} \vec{u}$ and $\vec{x} = \vec{u} - \vec{y}$.]

Solution. (a) Let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. Then $V^\perp$ is the null space of the matrix A^T, so, to find a basis for $V^\perp$ we first row reduce A^T.

$$A^T = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

So the general solution to the equation $A^T\vec{x} = \vec{0}$ is $x_1 = 2s-2t$, $x_2 = -2s+t$, $x_3 = s$, $x_4 = t$. Setting $s = 1$, $t = 0$ and then $s = 0$, $t = 1$ we get the basis

$$Y = \left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

for the null space of A^T. Thus, Y is a basis for $V^\perp$.

(b)

$$X' = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} \right)$$

(c) $Y' = \left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right)$

(d) X' is orthogonal to Y'x, so since each of X' and Y' is orthogonal, $X' \cup Y'$ is an orthogonal set in $\mathbb{R}^4$. Since $X' \cup Y'$ is orthogonal, it is linearly independent. Finally, since $X' \cup Y'$ is linearly independent and contains 4 vectors, it is a basis of $\mathbb{R}^4$.

(e) $\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

41. Let $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$, where

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 3 \end{bmatrix},$$

and let $V = \text{Span}(X)$.

(a) Use the Gram-Schmidt process *on the set* X to produce an orthogonal basis, $X' = (\vec{v}_1', \vec{v}_2', \vec{v}_3')$, for V.

(b) Find a basis, $Y = (\vec{w}_1)$, for the orthogonal complement of V in $\mathbb{R}^4$.

(c) Explain why $X' \cup Y = (\vec{v}_1', \vec{v}_2', \vec{v}_3', \vec{w})$ is an orthogonal basis for $\mathbb{R}^4$. (This doesn't require any further calculations.)

(d) Write the vector $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 6 \end{bmatrix}$ as a sum of two vectors, $\vec{u} = \vec{x} + \vec{y}$, where $\vec{x}$ is in V and $\vec{y}$ is in $V^{\perp}$. [*Hint.* Use $\vec{x} = \text{proj}_V \vec{u}$ and $\vec{y} = \vec{u} - \vec{x}$, or $\vec{y} = \text{proj}_{V^{\perp}} \vec{u}$ and $\vec{x} = \vec{u} - \vec{y}$.]

SOLUTION. (a) $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$, so $\{\vec{v}_1, \vec{v}_2\}$ is a basis for V and $X' = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{9} \\ 0 \\ -\frac{4}{9} \\ \frac{1}{9} \end{bmatrix} \right\}$ is an orthogonal basis for V.

(b) $Y = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) X' is orthogonal to Y, so since each of them is orthogonal, $X' \cup Y$ is an orthogonal set in $\mathbb{R}^4$. Since $X' \cup Y$ is orthogonal, it is linearly independent. Finally, since $X' \cup Y$ is linearly independent and contains 4 vectors, it is a basis of $\mathbb{R}^4$.

(d) $\vec{x} = \begin{bmatrix} \frac{9}{2} \\ 0 \\ 1 \\ \frac{9}{2} \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -\frac{3}{2} \\ 4 \\ 0 \\ \frac{3}{2} \end{bmatrix}$

42. Let $\vec{v}_1 = (2, -2, 1)$, $\vec{v}_2 = (2, 1, -2)$, and $\vec{v}_3 = (1, 2, 2)$.

(a) Show that $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is an orthogonal basis for $\mathbb{R}^3$.

(b) Write $\vec{u} = (-1, 0, 2)$ as a linear combination of the vectors in X. (Note that this is different from asking for the coordinate vector of $\vec{u}$ with respect to the basis X! The steps in the solution are the same but the form of the answer is different.)

(c) Turn X into an orthonormal basis, Y, of $\mathbb{R}^3$.

SOLUTION. (a) $\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, which shows that X is an orthogonal set in $\mathbb{R}^3$. Since X is an orthogonal set, X is linearly independent. Now, since X is linearly independent and contains 3 vectors, X is a basis of $\mathbb{R}^3$.

(b) $\vec{u} = -\frac{2}{3}\vec{v}_2 + \frac{1}{3}\vec{v}_3$

(c) $Y = \left\{ \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \right\}$

43. Let $\vec{u}_1 = (2, 2, -1)$, $\vec{u}_2 = (4, 1, 1)$, and $\vec{u}_3 = (1, 10, -5)$.

(a) Show that $X = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathbb{R}^3$.

(b) Apply the Gram-Schmidt process to this basis to find an orthogonal basis, X', of $\mathbb{R}^3$.

(c) Find the coordinate vector of $\vec{w} = (4, 6, 0)$ with respect to the basis X'.

(d) Further, turn X' into an orthonormal basis, X'', of $\mathbb{R}^3$.

(e) Find the coordinate vector of $\vec{w} = (4, 6, 0)$ with respect to the basis X''.

SOLUTION. (a) Rank $\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = 3$, so X is a basis of $\mathbb{R}^3$.

(b) $Y = \left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

(c) $Z = \left\{ \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \right\}$

44. Let $\vec{u}_1 = (0, 2, 1, 0)$, $\vec{u}_2 = (1, -1, 0, 0)$, $\vec{u}_3 = (1, 2, 0, -1)$ and $\vec{u}_4 = (1, 0, 0, 1)$.

(a) Show that $X = (\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ is a basis for $\mathbb{R}^4$.

(b) Apply the Gram-Schmidt process to this basis to find an orthogonal basis, X', of $\mathbb{R}^4$.

(c) Find the coordinate vector of $\vec{w} = (0, 5, 2, 5)$ with respect to the basis X'.

(d) Further, turn X' into an orthonormal basis, X'', of $\mathbb{R}^4$.

(e) Find the coordinate vector of $\vec{w} = (0, 5, 2, 5)$ with respect to the basis X''.

SOLUTION.

(a) Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Row reducing A we find that

$$A \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

This is far enought to see that the rank of A is 4. Thus, A is invertible, so X is a basis of $\mathbb{R}^4$.

(b) Applying the Gram-Schmidt procedure to X we get

$$\vec{u}_1' = \vec{u}_1 = (0, 2, 1, 0)$$

$$\begin{aligned}
\vec{u}_2' &= \vec{u}_2 - \left(\frac{\vec{u}_2 \bullet \vec{u}_1'}{\vec{u}_1' \bullet \vec{u}_1'}\right)\vec{u}_1' \\
&= (1,-1,0,0) - \left(\frac{0-2+0+0}{0+4+1+0}\right)(0,2,1,0) \\
&= (1,-1,0,0) + \frac{2}{5}(0,2,1,0) \\
&= \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3' &= \vec{u}_3 - \left(\frac{\vec{u}_3 \bullet \vec{u}_1'}{\vec{u}_1' \bullet \vec{u}_1'}\right)\vec{u}_1' - \left(\frac{\vec{u}_3 \bullet \vec{u}_2'}{\vec{u}_2' \bullet \vec{u}_2'}\right)\vec{u}_2' \\
&= (1,2,0,-1) - \left(\frac{0+4+0+0}{0+4+1+0}\right)(0,2,1,0) \\
&\quad - \left(\frac{1-\frac{2}{5}+0+0}{1+\frac{1}{25}+\frac{4}{25}}\right)\left(1,-\frac{1}{5},\frac{2}{5},0\right) \\
&= (1,2,0,-1) - \frac{4}{5}(0,2,1,0) - \frac{1}{2}\left(1,-\frac{1}{5},\frac{2}{5},0\right) \\
&= \left(\frac{1}{2},\frac{1}{2},-1,-1\right)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_4' &= \vec{u}_4 - \left(\frac{\vec{u}_4 \bullet \vec{u}_1'}{\vec{u}_1' \bullet \vec{u}_1'}\right)\vec{u}_1' - \left(\frac{\vec{u}_4 \bullet \vec{u}_2'}{\vec{u}_2' \bullet \vec{u}_2'}\right)\vec{u}_2' - \left(\frac{\vec{u}_4 \bullet \vec{u}_3'}{\vec{u}_3' \bullet \vec{u}_3'}\right)\vec{u}_3' \\
&= (1,0,0,1) - \left(\frac{0+0+0+0}{0+4+1+0}\right)(0,2,1,0) \\
&\quad - \left(\frac{1+0+0+0}{1+\frac{1}{25}+\frac{4}{25}+0}\right)\left(1,-\frac{1}{5},\frac{2}{5},0\right) \\
&\quad - \left(\frac{\frac{1}{2}+0+0-1}{\frac{1}{4}+\frac{1}{4}+1+1}\right)\left(\frac{1}{2},\frac{1}{2},-1,-1\right) \\
&= (1,0,0,1) - 0(0,2,1,0) - \frac{5}{6}\left(1,-\frac{1}{5},\frac{2}{5},0\right) + \frac{1}{5}\left(\frac{1}{2},\frac{1}{2},-1,-1\right) \\
&= \left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)
\end{aligned}$$

So $Y = \left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{4}{15} \\ \frac{4}{15} \\ -\frac{8}{15} \\ \frac{4}{5} \end{bmatrix} \right)$ is an orthogonal basis for $\mathbb{R}^4$.

(c) Finally, dividing each of the vectors in Y by its length gives the orthonormal basis

$$Z = \left(\begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{\sqrt{30}} \\ -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{2}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix} \right)$$