Math 375 (Complex Analysis) Your name: KEY

Quiz 7

April 18, 2012

1. State the definition of a limit of a sequence.

$$\lim_{n\to\infty} a_n = a \iff \forall \varepsilon>0\ \exists N>0\ \forall n\geq N:\ |a_n-a|<\varepsilon$$

2. For each of the sequences, determine convergence/divergence. If the sequence converges, find the limit.

(a) $\{\exp(i\pi/n)\}_{n\geq 1}$ Since e^z is continuous everywhere,

$$\lim_{n\to\infty} e^{\frac{i\pi}{n}} = e^{\lim_{n\to\infty}\frac{i\pi}{n}} = e^0 = 1$$

(b) $\left\{\frac{(-i)^n}{n^2}\right\}_{n\geq 1}$ $\lim_{n\to\infty}\left|\frac{(-i)^n}{n^2}\right| = \lim_{n\to\infty}\frac{1}{n^2} = 0 \Rightarrow \lim_{n\to\infty}\frac{(-i)^n}{n^2} = 0$

(c) $\left\{\frac{(-i+1)^n}{i+1}\right\}_{n\geq 1}$ $\left|\frac{(-i+1)^n}{i+1}\right| = \frac{(\sqrt{2})^n}{\sqrt{2}} = (\sqrt{2})^{n-1}$ is unbounded, so the sequence is divergent.

3. Determine, which of the following series are convergent. Explain your answer.

(a) $\sum_{k=0}^{\infty}\frac{2^k(k^2-4)}{k!}$ We apply ratio test:

$$\lim_{k\to\infty}\left|\frac{\frac{2^{k+1}((k+1)^2-4)}{(k+1)!}}{\frac{2^k(k^2-4)}{k!}}\right| = \lim_{k\to\infty}\left|\frac{2}{(k+1)}\cdot\frac{k^2+2k-3}{k^2-4}\right| = 0 < 1 \xrightarrow[\text{test}]{\text{ratio}}$$

$$\Rightarrow \sum_{k=0}^{\infty}\frac{2^k(k^2-4)}{k!} \text{ is convergent.}$$

(b) $\sum_{k=0}^{\infty}\left(\frac{1-2i}{\sqrt{3}}\right)^k$ This is a geometric series with

$\left|\frac{1-2i}{\sqrt{3}}\right| = \sqrt{\frac{1}{3}+\frac{4}{3}} = \sqrt{\frac{5}{3}} > 1$, so it must be divergent.