Math 375 (Complex Analysis) Your name: KEY

Quiz 6

March 29, 2012

1. State the First Fundamental Theorem of Calculus for functions of complex variables. (Hint: about the existence of ...)

If $f: G \to \mathbb{C}$ is a continuous function satisfying $\int_\gamma f(z)\,dz = 0$ for each closed curve $\gamma \subset G$, then the function $F(z) = \int_{\gamma_z} f(z)\,dz$, where γ_z is an arbitrary curve connecting a fixed pt. z_0 to z, is an antiderivative of f in G. I.e. $F'(z) = f(z)\ \forall z \in G$

2. Compute $\int_\gamma \frac{1}{z^2}\,dz$ where $\gamma(t) = t + i(t^2-1), t \in [-1,2]$ is a piece of parabola. Justify your answer.

It is straightforward to verify that $F(z) = \frac{-1}{z}$ is an antiderivative of $f(z) = \frac{1}{z^2}$ on $\mathbb{C} - \{0\}$.

Therefore,

$$\int_\gamma \frac{dz}{z^2} = \left(-\frac{1}{z}\right)\Bigg|_{z=-1=\gamma(-1)}^{z=2+3i=\gamma(2)} = -\frac{1}{2+3i} - \left(-\frac{1}{-1}\right) = \boxed{-\frac{15}{13} + \frac{3}{13}i}$$

3. Find a harmonic conjugate of a function $u(x,y) = x^2 + x - y^2$, i.e. a function $v(x,y)$ such that $u + iv$ is holomorphic.

Function v has to satisfy

$$\left.\begin{array}{l} v_x = -u_y = 2y \\ v_y = u_x = 2x+1 \end{array}\right\} \Rightarrow v = \int v_x\,dx + C(y) = \int 2y\,dx + C(y) = 2xy + C(y)$$

So

$$v_y = 2x + C'(y) = 2x + 1 \Rightarrow C'(y) = 1$$

and $C(y) = y + C$.

Thus, $\boxed{v(x,y) = 2xy + y + C}$