MATH 375 Complex Analysis Midterm March 22, 2012

1. (20 points) **Basics**

(1) Compute the following. State each of your answers in the form $x + iy$.

(a) $(2+3i)(-i+\pi) = -2i - 3i^2 + 2\pi + 3\pi i = (3+2\pi) + i(3\pi - 2)$

(b) $i - e^{i\frac{\pi}{2}} = i - i = 0$

(c) $|\cos 2 - i \sin 2| = \sqrt{\cos^2 2 + \sin^2 2} = 1$

(2) Show that $|z+w|^2 - |z-w|^2 = 4\,Re(z\overline{w})$ for any $z, w \in \mathbb{C}$.

$$(z+w)(\overline{z}+\overline{w}) - (z-w)(\overline{z}-\overline{w}) =$$

$$= \cancel{z\overline{z}} + w\overline{z} + \overline{w}z + \cancel{w\overline{w}} \cancel{-z\overline{z}} + z\overline{w} + \overline{z}w \cancel{- w\overline{w}} =$$

$$= 2z\overline{w} + 2\overline{z\overline{w}} = 4\,Re(z\overline{w})$$

2. (20 points) **Functions and Derivative**

(a) Give a definition of a holomorphic in a region G function .

A function f is holomorphic in a region G if it is differentiable at each point of G.

(b) Suppose that f is holomorphic on a region G and that $Re(f(z)) = 0$ for each $z \in G$. Prove that f is constant.

Suppose $f(z) = u(x,y) + iv(x,y)$. By Cauchy-Riemann equations $V_x(x,y) = -u_y(x,y) = 0$, $V_y(x,y) = u_x(x,y) = 0$ since $u(x,y) = 0$. Thus $V(x,y) = \text{const}$ and $f(z) = iV(x,y) = \text{const}$.

(c) Find the Möbius transformations satisfying $0 \mapsto 2 - i$, $1 \mapsto i$, $\infty \mapsto 1$. Write your answers in standard form $\frac{az+b}{cz+d}$

Suppose $f(z) = \frac{az+b}{cz+d}$

$f(0) = \frac{b}{d} = 2 - i \Rightarrow b = (2-i)d$

$f(\infty) = \frac{a}{c} = 1 \Rightarrow a = c$

$f(1) = \frac{a+b}{c+d} = \frac{c+(2-i)d}{c+d} = i \Rightarrow c + 2d - id = (c+d)i$

$(c+2d) + (-c-2d)i = 0 \Rightarrow c = -2d$.

Set $d = 1 \Rightarrow c = -2 \Rightarrow a = -2$, $b = (2-i)$. Thus

$f(z) = \frac{-2z + (2-i)}{-2z + 1}$

3. (20 points) Write **principal values** of the following expressions in the form $x + iy$.

(a) $(1-i)^{1+i} = \exp\left((1+i)\,\mathrm{Log}(1-i)\right) = \exp\left((1+i)\cdot\left(\ln\sqrt{2} + i\cdot\left(-\frac{\pi}{4}\right)\right)\right) =$

$= \exp\left(\left(\ln\sqrt{2} + \frac{\pi}{4}\right) + i\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right) =$

$= \sqrt{2}e^{\frac{\pi}{4}} \cdot \cos\left(\ln\sqrt{2} - \frac{\pi}{4}\right) + i \cdot \sqrt{2}e^{\frac{\pi}{4}} \sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)$

(b) $\log(-1 + \sqrt{3}) = \ln(-1+\sqrt{3}) + 0 \cdot i$

4. (20 points) **Integration**

(a) Compute $\int_\gamma \bar{z}\, dz$, where γ is a circle centered at i of radius 2 oriented clockwise.

$$\int_\gamma \bar{z}\, dz = \left| \begin{array}{l} \gamma(t) = i + 2e^{-it} \\ t \in [0, 2\pi] \end{array} \right| = \int_0^{2\pi} \overline{(i + 2e^{-it})} \cdot (-2i) e^{-it}\, dt =$$

$$= -\int_0^{2\pi} (-i + 2e^{+it}) \cdot 2i e^{-it}\, dt = \underbrace{-2\int_0^{2\pi} e^{-it}\, dt}_{= 0} - 4i \int_0^{2\pi} dt =$$

$$= -8\pi i$$

(b) Use the definition of a length to find the length of a curve parameterized by $\gamma(t) = -t + it^2$, $t \in [-1, 2]$.

$$\text{length}(\gamma) = \int_{-1}^{2} |\gamma'(t)|\, dt = \int_{-1}^{2} |-1 + 2ti|\, dt = \int_{-1}^{2} \sqrt{1 + 4t^2}\, dt =$$

$$= \left| \begin{array}{l} 2t = \tan u \\ dt = \frac{1}{2\cos^2 u} du \\ \begin{array}{c|c} t & u \\ \hline -1 & \arctan(-2) \\ 2 & \arctan 4 \end{array} \end{array} \right| = \int_{\arctan(-2)}^{\arctan 4} \frac{1}{\cos u} \cdot \frac{1}{2\cos u}\, du = \frac{1}{2} \int_{\arctan(-2)}^{\arctan 4} \sec^3 u\, du \;=$$

$$\int \sec^3 u\, du = \int \sec u \cdot \sec^2 u\, du = \left| \begin{array}{ll} w = \sec u & dw = \sec u \cdot \tan u\, du \\ dv = \sec^2 u\, du & v = \tan u \end{array} \right| =$$

$$= \sec u \cdot \tan u - \int \tan^2 u \cdot \sec u\, du = \left| \tan^2 u = \sec^2 u - 1 \right| =$$

$$= \sec u \cdot \tan u - \int \sec^3 u\, du + \int \sec u\, du. \quad \text{Thus}$$

$$\int \sec^3 u\, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \int \sec u\, du = \frac{1}{2} \sec u \cdot \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C$$

$$= \frac{1}{4} \left[\sec(\arctan 4) \tan(\arctan(4)) + \ln|\sec(\arctan 4) + 4| - \sec(\arctan(-2)) \cdot (-2) - \ln|\sec(\arctan(-2)) - 2| \right]$$

$$= \frac{1}{4} \left[2\sqrt{5} - \ln(-2 + \sqrt{5}) + 2\sqrt{17} - \ln(-4 + \sqrt{17}) \right]$$

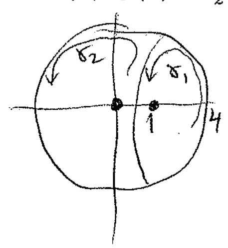
5. (20 points) **Cauchy's theorem and Consequences**
Integrate the following functions over the circle $\{z \in \mathbb{C} \colon |z| = 4\}$ oriented counterclockwise.

(a) $f(z) = \exp(z)\sin(z^2)$

$f(z)$ is holomorphic in the region, so

$$\int_\gamma f(z)\,dz = 0$$

(b) $g(z) = \frac{1}{z^2(z-1)}$

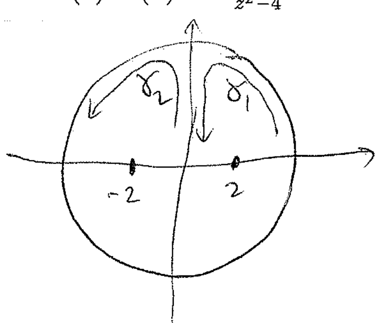


$$\int_\gamma g(z)\,dz = \int_{\gamma_1} \frac{\frac{1}{z^2}}{z-1}\,dz + \int_{\gamma_2} \frac{\frac{1}{z-1}}{z^2}\,dz =$$

$$= 2\pi i \cdot \frac{1}{1^2} + 2\pi i \cdot \frac{d}{dz}\left(\frac{1}{z-1}\right)\Big|_{z=0} =$$

$$= 2\pi i \cdot \left(1 + \frac{-1}{(z-1)^2}\Big|_{z=0}\right) = 2\pi i(1-1) = 0$$

(c) $h(z) = \frac{\cos(z+i)}{z^2-4}$



$$\int_\gamma h(z)\,dz = \int_{\gamma_1} \frac{\frac{\cos(z+i)}{z+2}}{z-2}\,dz + \int_{\gamma_2} \frac{\frac{\cos(z+i)}{z-2}}{z+2}\,dz =$$

$$= 2\pi i\left(\frac{\cos(z+i)}{z+2}\Big|_{z=2} + \frac{\cos(z+i)}{z-2}\Big|_{z=-2}\right) =$$

$$= 2\pi i\left(\frac{1}{4}\cos(2+i) - \frac{1}{4}\cos(-2+i)\right) =$$

$$= \frac{\pi i}{2}\left(\cos(2+i) - \cos(-2+i)\right)$$