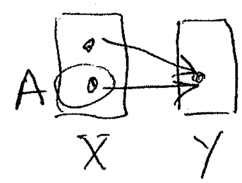
# Review problems key
# (Topology I, MATH 461, 10/20/11)

(1) $x \in A \Rightarrow f(x) \in f(A) \Rightarrow x \in f^{-1}(f(A))$.
Converse does not hold:




(2). Let $f: X \to Y$ be s.t. $f(x) = a$ for some $a \in Y$ and all $x \in X$.
Then $\forall O$-open in $Y$ $f^{-1}(O) = \begin{cases} \emptyset, & a \notin O \\ X, & a \in O \end{cases}$ — open in $X$.

(3) $\{\emptyset, X\}$, $\{\emptyset, \{x\}, X\}$, $\{\emptyset, \{y\}, X\}$, $\{\emptyset, \{z\}, X\}$,
$\{\emptyset, \{x,y\}, X\}$, $\{\emptyset, \{x,z\}, X\}$, $\{\emptyset, \{y,z\}, X\}$,
$\{\emptyset, \{x,y\}, \{z\}, X\}$, $\{\emptyset, \{x,z\}, \{y\}, X\}$, $\{\emptyset, \{y,z\}, \{x\}, X\}$,
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$\{\emptyset, \{z\}, \{y,z\}, X\}$, $\{\emptyset, \{x\}, \{y\}, \{x,y\}, X\}$, $\{\emptyset, \{x\}, \{z\}, \{x,z\}, X\}$, $\{\emptyset, \{y\}, \{z\}, \{y,z\}, X\}$.
Fineness is defined by inclusion

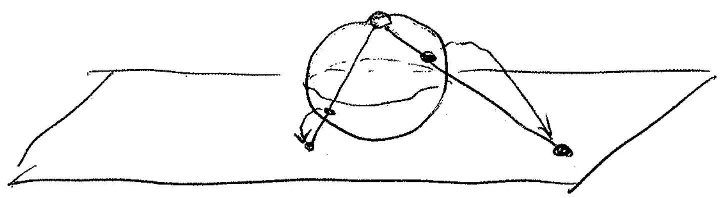
(4) Use Proposition 1.2 from Hatcher.
Note: in (c) $B_n'$s are disjoint, so (2) in Prop. 1.2 is satisfied automatically
in (b) $S$ could be empty. Use this to check condition (1) in Prop. 1.2

(5) $f^{-1}((\tfrac{1}{2},\tfrac{3}{2})) = \mathbb{R}-\mathbb{Q}$ – not open in $\mathbb{R}$.

(6) Use linear function $f(x) = cx + d$. Pick c and d such that $f(0)=a$, $f(1)=d$.

(7) $f(x) = 1 - \ln x$

(8) This is called stereographic projection:



(9) $B_1(0) \underbrace{\cong \mathbb{R}^2 \cong}_{\text{homework}} (0,1)\times(0,1)$

(10) (a) $(x,y_1) \sim (x,y_2)$ $\forall\, y_1, y_2 \in \mathbb{R}$

(b) $(x,y_1) \sim (x+k, y_2)$ $\forall\, y_1, y_2 \in \mathbb{R}$, $k \in \mathbb{Z}$.

(c) $(x,y) \sim (x+k, y)$ $\forall\, k \in \mathbb{Z}$

(11) (a) – Yes

(b) – NO – a bit tricky. Use the fact that if both coords are rational, or both coords are irrational, then we are not in this set.

(c) – YES. It is path-connected.

(12) No: $S^1$ is compact in $\mathbb{R}^2$.

(13) Yes: Use prop. on page 24 in Hatcher and induct on n.

(14) Use Heine-Borel thm.

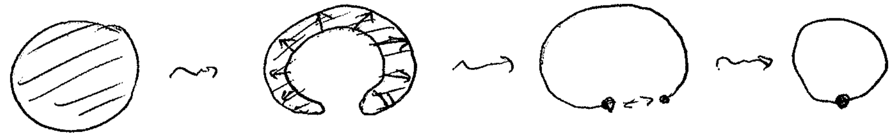
(15) Klein bottle is a quotient of $[0,1]\times[0,1]$.

(16) Either use a definition and explicitly construct a subcover, ~~a~~ or argue with Heine-Borel thm.

(17) $X=\{1,2\}$, $\mathcal{O}=\{\emptyset, \{1\}, \{1,2\}\}$. Then $\{1\}$ is compact, but not closed.

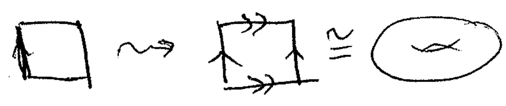
Hatcher:

(4.2) Just use a definition of a quotient map

(4.3) $D^2 \to S^1$: 

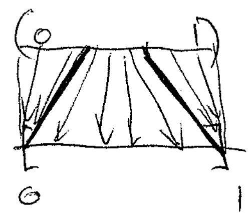
$S^2 \to D^2$: 

$S^2 \to S^1$ — composition of previous 2

$S^2 \to S^1 \times S^1$: $S^2 \to D^2 \cong$ 

$S^1 \times S^1 \to S^2$: look at Torus wiki page

Another: 

(4.4) (a) 

$[0,1]$ — compact, so any quotient space will be compact.

(b) $C \to [0,1]$: $C \cong \{x_1x_2x_3\ldots \mid x_i \in \{0,2\}\}$. Replace 2 with 1, and then $0.x_1\ldots x_k0111\ldots \sim 0.x_1x_2\ldots x_k100\ldots$

$[0,1]$ is connected $\Rightarrow$ any quotient will be connected.