Math 461 (Topology I) Fall 2011

Review problems for the final

Note: I will not collect this assignment – just do it for your benefit. This is a preparational homework for the final.

Solve the following problems

1. Let $X \subset \mathbb{R}^3$ be the union of the spheres of radius 2 centered at $(0,0,0)$ and $(3,0,0)$. Draw X and draw a simplicial complex whose underlying space is homeomorphic to X. Compute the Euler characteristic of your complex.

2. Give an explicit homeomorphism between $\mathbb{R}^2$ and the cone $\{(x,y,z) \in R^3 | x^2+y^2=z^2, z \geq 0\}$.

3. A space X is *locally compact* at a point $x \in X$ if x has an open neighborhood which itself has a compact neighborhood. We say that X is *locally compact* if it is locally compact at every point.

   (a) Prove a compact space is locally compact.

   (b) Prove R^n is locally compact.

   (c) Prove a punctured surface with boundary is locally compact.

   (d) Is $X := interior(D^2) \cup \{(1,0)\}$ locally compact?

4. Prove that if X is a compact space, then every sequence $x_1, x_2, x_3, \dots \in X$ has a cluster point – that is, there is a point $x \in X$ such that every neighborhood of x contains x_n for infinitely many n.

5. Consider the subset of $\mathbb{R}^2$ defined by $T_n := \{(x,y) \in \mathbb{R}^2 | x = \frac{1}{n} \text{ and } 0 \leq y \leq 1\}$, for any integer n, where for the purposes of this problem $T_0 := \{(x,y) \in \mathbb{R}^2 | x = 0 \text{ and } 0 \leq y \leq 1\}$. Finally, let $I := \{(x,y) | y = 0 \text{ and } 0 \leq x \leq 1\}$. The topologist's comb is the union $T := I \cup \bigcup_{i \geq 0} T_i$. For each the following subspaces of R^2, determine which of the following properties it possesses: (i) closed in R^2; (ii) compact; (iii) locally compact; (iv) connected; (v) path-connected; (vi) open in R^2. Justify your response.

   (a) The toplogist's comb T;

   (b) The topologist's comb missing a tooth, $M := T - T_0$;

   (c) The broken topologist's comb, $B := T - \{(0,0)\}$.

6. Let $M := \mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{P}^2 \# \mathbb{K}^2$.

   (a) What is the Euler characteristic of M?

   (b) How is M listed in the classification?

   (c) Give a polygonal disk with gluing scheme such that the quotient space is homeomorphic to M.

7. Show that there is a 2-sheeted covering $T^2 \to K$ of the Klein bottle by the torus.

8. For some index set I, for every $i \in I$ let $A_i \subset \mathbb{R}^n$ be compact. Prove that $\bigcap_{i \in I} A_i$ is compact.

9. Let $B \subset \mathbb{R}^2$ be a disk, and let $j : \partial B \to \partial B$ be a homeomorphism. Show there is a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H|_{\partial B} = h$.

10. Suppose G is a group for which $x^2 = e$ for each $x \in G$. Show that $ab = ba$ for all $a, b \in G$ (Such groups are called *abelian*).

11. Let X be a topological space consisting of n points with discrete topology. How many elements are there in the set of homotopy classes of functions from X to

    (a) $\mathbb{R}$

    (b) $\mathbb{R} - \{0\}$

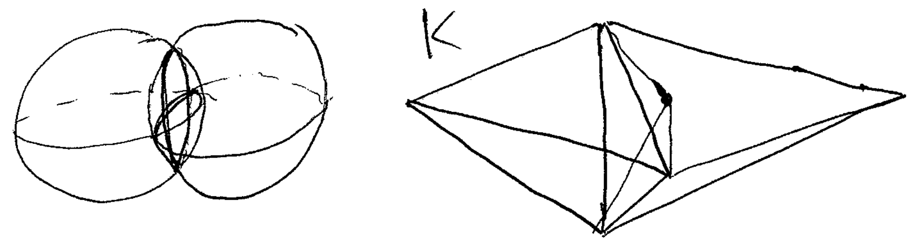
    (c) $\mathbb{R} - \{0, 1\}$

    (d) $\mathbb{R}^2 - \{(0, 0)\}$

12. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. Prove that if f does not have fixed points, then f is homotopic to the central symmetry $g(x) = -x$.

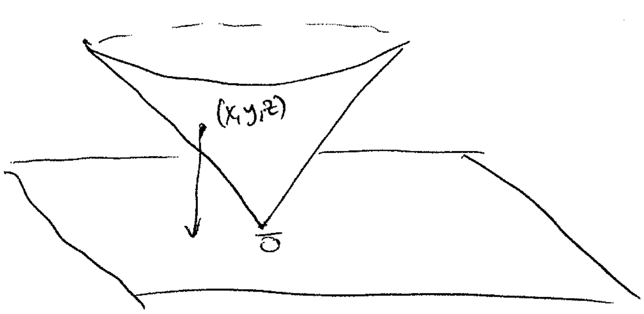
Review problem answers and some solutions.
(Topology 1, Fall 2011)
Dmytro Savchuk

(1)



$f_0(K) = 6$
$f_1(K) = 12 \Rightarrow \chi(K) = 6 - 12 + 10 = 4$
$f_2(K) = 10$

(2)

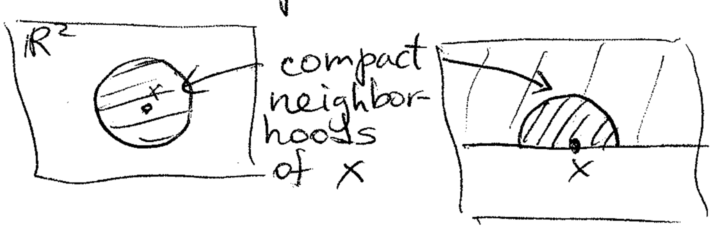


$(x,y,z) \longrightarrow (x,y)$

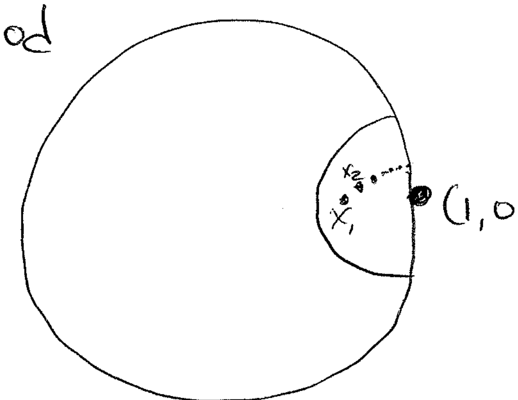
(3) (a) Any point $x \in X$ has an open neighborhood X that has compact neighborhood X.

(b) $\forall x \in \mathbb{R}^n \; \exists \; B_1(x)$ – open nbhd of x s.t. $\exists$ $\overline{B_1(x)}$ – compact neighborhood of $B_1(x)$

(c) Each point of a punctured surface w/boundary has a nbhd $\cong \mathbb{R}^2$ or $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. Both of these spaces are locally compact

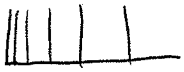
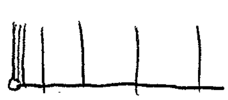
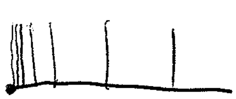


(d) Each open neighborhood of $(1,0)$ will contain a sequence of points without a condensation point in X. So it cannot be surrounded by a compact neighborhood



(4) Suppose not. Then $\forall x \in X$ let U_x be an open nbhd of x that contains only finitely many x_i's. Then $\{U_x, x \in X\}$ is an open cover, that by compactness of X will have finite subcover. But then X is a union of finite number of sets containing finite number of x_i's. This contradicts to the fact that there are infinitely many x_i's.

(5)

		closed	compact	loc. compact	connected	path connected
(a)		Y	Y	Y	Y	Y
(b)		N	N	Y	Y	Y
(c)		N	N	Y	Y	N
(d)		N	N	N	Y	Y

$(T-T_0) \cup \{(0,0)\}$

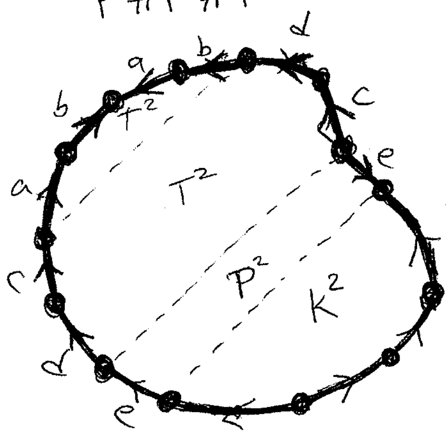
(6) (a) $\chi(M) = \chi(T^2)+\chi(T^2)+\chi(P^2)+\chi(K^2)-6 =$

$= 0+0+1+0-6 = -5$

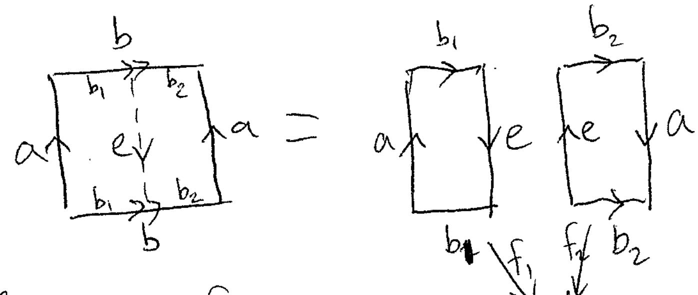
(b) $M = T^2 \# \underbrace{T^2 \# P^2}_{P^2\#P^2\#P^2} \# \underbrace{P^2 \# P^2}_{K^2} = \underbrace{P^2 \# \dots \# P^2}_{7 \text{ times}}$

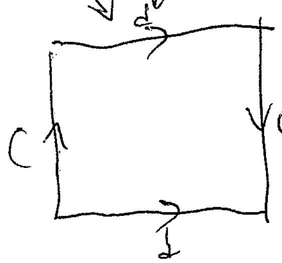
$\underbrace{T^2 \#}_{P^2\#P^2\#P^2}$

(c)



(7) Torus



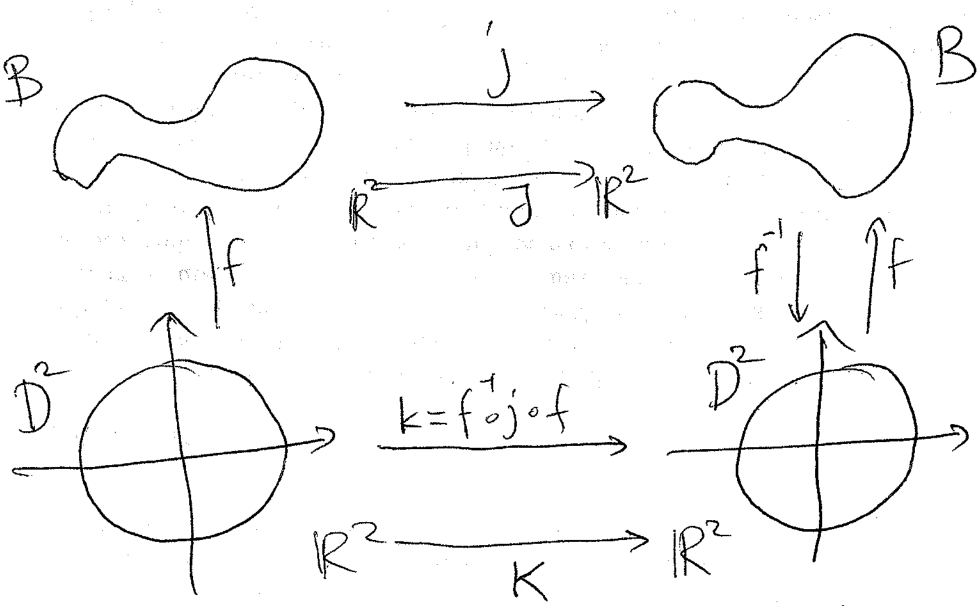


Maps f_1, f_2 from halves of torus are defined just by stretching the rectangle to the square. Then one has to use pasting lemma to glue f_1, f_2 to $f: T^2 \to K^2$.

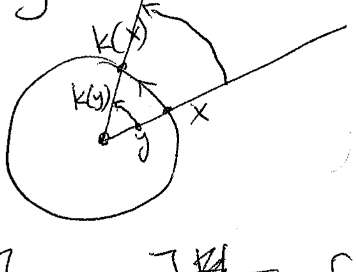
Then check that f is indeed a 2-sheeted covering.

8. Intersection of arbitrary family of closed subsets of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$. It is also bounded since each A_i is bounded. Thus, by Heine-Borel Theorem, it must be compact.

9. Suppose we want to extend homeo $j : \partial B \to \partial B$. By Schnoflies theorem $\exists\ f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(D^2) = B$ and $f(\partial D^2) = \partial B$



This homeo f induces a homeomorphism $k = f^{-1} \circ j \circ f : \partial D^2 \to \partial D^2$. We extend k to $K : \mathbb{R}^2 \to \mathbb{R}^2$ by "rotating" the rays from the origin according to k: $\forall y \in \mathbb{R}^2$ define



$$K(y) = \begin{cases} \|y\| \cdot k\left(\frac{y}{\|y\|}\right), & y \neq 0 \\ 0, & y = 0. \end{cases}$$

Then $J = f \circ K \circ f^{-1}$ is the desired homeomorphism.

10. For any $a, b \in G$ we have

$$e = (ab)^2 = abab \implies (ab)^{-1} = (ab)^{-1}(ab)(ab) = ab$$

But $(ab)^{-1} = b^{-1}a^{-1} = ba$ since $a^2 = b^2 = e$. So

$ab = ba$.

11. (a) 1 (any two maps are homotopic)

(b) 2^n (each point in X goes to one of two path connected components)

(c) 3^n (—— " —— three
—— " ——)

(d) 1 (since $\mathbb{R}^2 - \{(0,0)\}$ is path connected)

12. Let $f: S^n \to S^n$ be a continuous map with no fixed points. Let $g(x) = -x$.

Define homotopy as

$$H(x,t) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}$$

Since $f(x) \neq x$, we have $(1-t)f(x) + t(-x) \neq 0$.

Indeed, if $t \neq \frac{1}{2}$, then $\|(1-t)f(x)\| = |1-t| \neq |t| = \|t(-x)\|$,

and if $t = \frac{1}{2}$, then $\frac{1}{2}f(x) - \frac{1}{2}x \neq 0$.

Thus H is continuous.