Math 461 (Topology I) Fall 2011

Review problems for the final

Note: I will not collect this assignment – just do it for your benefit. This is a preparational homework for the final.

Solve the following problems

1. Let $X \subset \mathbb{R}^3$ be the union of the spheres of radius 2 centered at $(0,0,0)$ and $(3,0,0)$. Draw X and draw a simplicial complex whose underlying space is homeomorphic to X. Compute the Euler characteristic of your complex.

2. Give an explicit homeomorphism between $\mathbb{R}^2$ and the cone $\{(x,y,z) \in R^3 | x^2+y^2 = z^2, z \geq 0\}$.

3. A space X is *locally compact* at a point $x \in X$ if x has an open neighborhood which itself has a compact neighborhood. We say that X is *locally compact* if it is locally compact at every point.

   (a) Prove a compact space is locally compact.

   (b) Prove R^n is locally compact.

   (c) Prove a punctured surface with boundary is locally compact.

   (d) Is $X := interior(D^2) \cup \{(1,0)\}$ locally compact?

4. Prove that if X is a compact space, then every sequence $x_1, x_2, x_3, \dots \in X$ has a cluster point – that is, there is a point $x \in X$ such that every neighborhood of x contains x_n for infinitely many n.

5. Consider the subset of $\mathbb{R}^2$ defined by $T_n := \{(x,y) \in \mathbb{R}^2 | x = \frac{1}{n} \text{ and } 0 \leq y \leq 1\}$, for any integer n, where for the purposes of this problem $T_0 := \{(x,y) \in \mathbb{R}^2 | x = 0 \text{ and } 0 \leq y \leq 1\}$. Finally, let $I := \{(x,y) | y = 0 \text{ and } 0 \leq x \leq 1\}$. The topologist's comb is the union $T := I \cup \bigcup_{i \geq 0} T_i$. For each the following subspaces of R^2, determine which of the following properties it possesses: (i) closed in R^2; (ii) compact; (iii) locally compact; (iv) connected; (v) path-connected; (vi) open in R^2. Justify your response.

   (a) The toplogist's comb T;

   (b) The topologist's comb missing a tooth, $M := T - T_0$;

   (c) The broken topologist's comb, $B := T - \{(0,0)\}$.

6. Let $M := \mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{P}^2 \# \mathbb{K}^2$.

   (a) What is the Euler characteristic of M?

   (b) How is M listed in the classification?

   (c) Give a polygonal disk with gluing scheme such that the quotient space is homeomorphic to M.

7. Show that there is a 2-sheeted covering $T^2 \to K$ of the Klein bottle by the torus.

8. For some index set I, for every $i \in I$ let $A_i \subset \mathbb{R}^n$ be compact. Prove that $\cap_{i\in I} A_i$ is compact.

9. Let $B \subset \mathbb{R}^2$ be a disk, and let $j : \partial B \to \partial B$ be a homeomorphism. Show there is a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H|_{\partial B} = h$.

10. Suppose G is a group for which $x^2 = e$ for each $x \in G$. Show that $ab = ba$ for all $a, b \in G$ (Such groups are called *abelian*).

11. Let X be a topological space consisting of n points with discrete topology. How many elements are there in the set of homotopy classes of functions from X to

    (a) $\mathbb{R}$

    (b) $\mathbb{R} - \{0\}$

    (c) $\mathbb{R} - \{0, 1\}$

    (d) $\mathbb{R}^2 - \{(0, 0)\}$

12. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. Prove that if f does not have fixed points, then f is homotopic to the central symmetry $g(x) = -x$.