Math 461 (Topology I) Fall 2011

Review problems for the midterm

Note: I will not collect this assignment – just do it for your benefit. This is a preparational homework for the midterm that covers the topics that will be presented on the midterm. The midterm will be over Chapters 2-6 in the text.

Solve the following problems

1. Let $f\colon X \to Y$ be a function. Prove that $A \subset f^{-1}(f(A))$. Does converse always hold?

2. Prove that any constant map is continuous.

3. Find all possible topologies on $\{x, y, z\}$. Order these topologies as to fineness and coarseness.

4. Prove that each of the following are bases for topologies on the prescribed sets:

   (a) the set of intervals of the form $[a, b)$ in $\mathbb{R}$.

   (b) $X = \{f : [0, 1] \to [0, 1]\}$ – the set of functions from $[0, 1]$ to itself. The collection of subsets of X of the form
   $$B_S = \{f \in X \mid f(x) = 0, x \in S\},$$
   where S is a subset of $[0, 1]$.

   (c) $X = \{p \mid p \text{ is a polynomial with real coefficients}\}$ and the collection of subsets of X of the form
   $$B_n = \{p \in X \mid \text{ degree of } p \text{ is equal to } n\}, \quad n \geq 0.$$

5. Consider a Dirichlet function $f\colon \mathbb{R} \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 0, & x \text{ is rational,} \\ 1, & \text{otherwise.} \end{cases}$$
   Is it continuous? Why?

6. Construct an explicit homeomorphism between open intervals $(0, 1)$ and (a, b) for arbitrary $a < b$.

7. Construct an explicit homeomorphism between open intervals $(0, 1)$ and $(1, \infty)$ for arbitrary $a < b$.

8. Construct an explicit homeomorphism between a 2-sphere with one point deleted and $\mathbb{R}^2$.

9. Construct an explicit homeomorphism between an open ball $B_1(0)$ in $\mathbb{R}^2$ and $(0, 1) \times (0, 1)$.

10. Find the equivalence relations on $\mathbb{R}^2$, such that the corresponding quotient spaces are homeomorphic to:

    (a) a line

    (b) a circle

    (c) an infinite cylinder

11. Which of the following subsets of $\mathbb{R}^2$ are connected?

    (a) $\{(x, y) \in \mathbb{R}^2 \mid y = \frac{x}{n}, n = 1, 2, 3, \ldots\}$

(b) $\{(x, y) \in \mathbb{R}^2 \mid \text{either } x \text{ or } y, \text{ but not both, is rational}\}$

(c) $\{(x, y) \in \mathbb{R}^2 \mid x \neq 1\}$

(d) $\{(x, y) \in \mathbb{R}^2 \mid x \neq 1\} \cup \{(0, 1)\}$.

12. Suppose that A is a compact subset of $\mathbb{R}^2$. Is $\mathbb{R}^2 - A$ always connected?

13. Suppose $X = \cup_{i=1}^{n} A_i$, where $A_i \cap A_{i+1} \neq \emptyset$, $i = 1, \ldots, n-1$ and each A_i is connected. Is X necessarily connected?

14. Is the intersection of two compact subspaces of $\mathbb{R}^n$ always compact?

15. Prove that the Klein bottle is a compact space.

16. Prove that the set $A = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ is a compact subspace of $\mathbb{R}$.

17. Construct a topological space X and a compact set A in X such that A is not closed. (Hint: think finite!)

    Hatcher's notes: 4.2, 4.3, 4.4.