

Review problems for the final with answers, hints, and some solutions

Note: I will not collect this assignment – just do it for your benefit. This is a preparational homework for the final that covers the topics that will be presented on the midterm. The final will be over Chapters 1-10 in the text with emphasis on Chapters 7-10 (“discrete” part of the course).

1. go over the review problems for the midterm (posted on my web-page), and problems from the last two homeworks.

Solve the following problems

2. Prove that for the doubling map D the point $x \in [0, 1]$ is eventually periodic if and only if it is rational.

Solution: If x is eventually periodic, then x can be represented as a sum of infinite geometric progression with rational first term and rational ratio. The sum of such progression is always rational.

On the other hand, if $x = \frac{p}{q}$ is rational, then the iterates of x under D will all have form $\frac{p'}{q}$ with $p' < q$. Since there are only finitely many such possibilities, there must be i and j such that $D^i(x) = D^j(x)$. Thus x is eventually periodic.

3. Let $f(x) = ax + b$ is a function from \mathbb{R} to \mathbb{R} . For which values of a and b does f have an attracting fixed point? A repelling fixed point?

Solution: First of all, there must be some fixed point. This is the case when $f(x) = ax + b = x$ has a solution, i.e. when $a \neq 1$ or $a = 1, b = 0$. In the first case there is exactly one fixed point which is attracting if $|a| < 1$ and repelling if $|a| > 1$. In the second case every point in \mathbb{R} is an L -stable fixed point, which is neither attracting nor repelling.

4. Let $f(x) = x + x^3$. Find all fixed points of f and decide whether they are attracting or repelling.

Solution: The only fixed point is 0. It is repelling since $f(x) = x^3 + x > x$ if $x > 0$ and $f(x) < x$ if $x < 0$. So in this case one can apply graphical analysis.

5. Give an example of a function $h(x)$ for which $h'(0) = 1$ and $x = 0$ is an attracting fixed point.

Solution: One can take $f(x) = \sin(x)$ and apply graphical analysis.

6. Draw a transition graph for a tripling map $T(x) = 3x(\text{mod } 1)$ and prove from this graph that there are points of arbitrary periods.

Solution: In the transition graph there are 3 nodes labelled by 0, 1 and 2, and each node is connected to each via an oriented edge (including the loops at each node). Therefore as an irreducible infinite periodic string realizing period n one can take $(1^{n-1}2)^\infty$.

7. Prove that the tripling map has sensitive dependence on initial conditions on $[0, 1]$.

Solution: The proof is analogous to the proof of the same statement for the doubling map (Theorem 10.4.4, page 391).

8. Let $G(x) = 4x(1 - x)$ be the logistic map. Prove that there are points in $[0, 1]$ that are not fixed points, periodic points, or eventually periodic points of G .

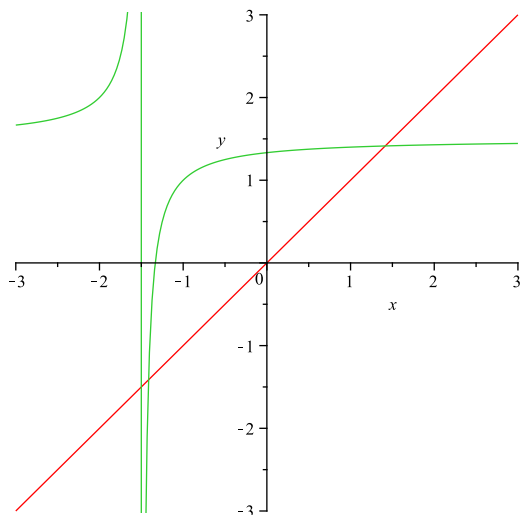


Figure 1: Graph of $f^2(x)$ in problem 9

Solution: There are several approaches. For example one can say that since $G(x)$ is conjugate to the tent map T and T is topologically transitive, G also has to be topologically transitive. Therefore there should be a point $x \in [0, 1]$ with a dense orbit. Such an orbit must be infinite, so x is not eventually periodic point (hence, x is neither fixed nor periodic).

Also one can use cardinality argument. There are just finitely many points of any given period ($G^n(x) = x$ has at most 2^n roots). So there can be only countably many preperiodic points in $[0, 1]$. Hence, there must be some points in $[0, 1]$ that are not preperiodic.

9. Define $x_{n+1} = \frac{x_n+2}{x_n+1}$.

(a) For $x_0 > 0$ find $L = \lim_{n \rightarrow \infty} x_n$.

(b) What negative points $x_0 < 0$ lie in the basin of attraction of L ?

Solution: We are iterating the function $f(x) = \frac{x+2}{x+1}$. The fixed points are $\pm\sqrt{2}$ and the graphical analysis of $f^2(x) = \frac{3x+4}{2x+3}$ (see the graph in Figure 1) shows that the basin of attraction of $L = \sqrt{2}$ is $\mathbb{R} - \{-\sqrt{2}\}$.

10. For the map $g(x) = 3.05x(1-x)$ find the stability of all fixed points and period-2 points.

Solution: The fixed points are 0 and 0.6721311475. Both of them are repelling as the $|g'(x)| > 1$ for both of them.

Period 2 points are 0.7377049180 and 0.5901639344, which form an attracting period-2 orbit.

11. Let $f : [0, \infty) \rightarrow [0, \infty)$ be a smooth (continuously differentiable) function, $f(0) = 0$, and let $p > 0$ be a fixed point such that $f'(p) \geq 0$. Assume further that $f'(x)$ is decreasing on $[0, \infty)$. Prove that all positive x converge to p under iterations of f . 1 *Solution:*

Solution: Since $f'(x)$ is decreasing, the graph of the function must be concave down. Therefore, the f is never decreasing (otherwise the graph would cross x -axis, which is impossible by definition of f). Also, the derivative at p must be in $(0,1)$ because if $f'(p) \geq 1$ then in order to satisfy $f(0) = 0$ the graph of f has to be concave up at some point. Therefore in our situation one can apply the graphical analysis to conclude that p is attracting with the basin $(0, \infty)$.

12. Write down a ternary expansion of $\frac{3}{5}$.

Solution: The trajectory of $\frac{3}{5}$ under iterations of the tripling map is

$$\frac{3}{5} \rightarrow \frac{4}{5} \rightarrow \frac{2}{5} \rightarrow \frac{1}{5} \rightarrow \frac{3}{5}.$$

We also have $\frac{3}{5} \in I_1$, $\frac{4}{5} \in I_2$, $\frac{2}{5} \in I_1$ and $\frac{1}{5} \in I_0$. Therefore the ternary expansion of $\frac{3}{5}$ is

$$0.(1210)^\infty.$$

13. For a tripling map T find a point x such that the orbit of x under T is dense in $[0, 1]$.

Solution: List all finite strings over $\{0, 1, 2\}$:

$$0, 1, 2, 00, 01, 02, 10, 11, 12, 20, 21, 22, 000, 001, \dots$$

and let ternary expansion of x be $0.012000102101112202122000001\dots$. Since any finite string over $\{0, 1, 2\}$ can be found in this expansion, the orbit of x comes arbitrarily close to any point in $[0, 1]$.

14. For a tripling map T find a point x which is not preperiodic and whose orbit under T is not dense in $[0, 1]$.

Answer: The orbit of irrational number $x = 0.2020020002\dots$ never visits the interval $[\frac{1}{3}, \frac{2}{3}]$ because there are no ones in the expansion for x , so it can't be dense.

15. How many attracting orbits can a function $f(x) = ax^2 + bx + c$ have?

Solution: Since the Schwarzian derivative of f is always negative, the number of attracting periodic orbits can not exceed the number of critical points of f , which is 1.

16. Which of the following sets are dense in $[0, 1]$?

- (a) the set of all points in $[0, 1]$ whose ternary expansions contain each digit infinitely many times;

Answer: Yes

- (b) the set of all points in $[0, 1]$ whose binary expansions contain some digit only finitely many times;

Answer: Yes. This is just a set of rational numbers

- (c) the set of all points in $[0, 1]$ whose ternary expansions do not contain digit 1;

Answer: No. This is a middle third Cantor set that does not contain points from $(\frac{1}{3}, \frac{2}{3})$.

- (d) $[0, 1] - X$, where X is arbitrary finite set;

Answer: Yes

- (e) the set of rational multiples of π .

Answer: Yes

17. Prove that a point $\frac{1}{4}$ belongs to the middle third Cantor set.

Solution: The ternary expansion of $\frac{1}{4}$ is $0.(02)^\infty$ and it does not involve any ones. Therefore, by characterization of the Cantor set, $\frac{1}{4}$ is one of its elements.

18. Is the tent map $T(x)$ expansive? Why?

Solution: No: for every $1 > r > 0$ one can provide points $x = \frac{1}{2} - \frac{r}{4}$ and $y = \frac{1}{2} + \frac{r}{4}$ such that $|x - y| = \frac{r}{2} < r$ and for any iteration $k > 0$ we have $|T^k(x) - T^k(y)| = 0$ (as images of x and y under T coincide).

19. For which values of a and b does the map $f(x) = ax + b$ have a sensible dependence on initial conditions?

Solution: For any x, y, a and b we have

$$|f^n(x) - f^n(y)| = |a|^n \cdot |x - y|.$$

Therefore, f has sensitive dependence on initial conditions if and only if $|a| > 1$.

20. Prove that for any polynomial $f : \mathbb{C} \rightarrow \mathbb{C}$ of degree at least 2 the Julia set of f is bounded.

Solution: For sufficiently large radius r the points outside the disk centered at 0 with radius r will escape to infinity (can be proven similarly to the $z^2 + c$ case: use that z^n is the biggest term if $|z|$ is big). Therefore the Julia set must belong to the above mentioned disk and hence needs to be bounded.

21. Does the orbit of the point $i + 1$ stay bounded under iterations of $f(z) = z^2 - 1$?

Solution: Since the first iteration $f(i + 1)$ of $i + 1$ is outside the threshold radius ($r(1) = \max\{1, 2\} = 2$ and $|(1 + i)^2 - 1| = |-1 + 2i| = \sqrt{5} > 2 = r(1)$), we must have that the iterates of x will converge to infinity.