Math 372 (Dynamical systems) Fall 2010
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Review problems for the final

Note: I will not collect this assignment – just do it for your benefit. This is a preparational homework for the final that covers the topics that will be presented on the midterm. The final will be over Chapters 1-10 in the text with emphasis on Chapters 7-10 ("discrete" part of the course).

1. go over the review problems for the midterm (posted on my web-page), and problems from the last two homeworks.

Solve the following problems

2. Prove that for the doubling map D the point $x \in [0,1]$ is eventually periodic if and only if it is rational.
3. Let $f(x) = ax + b$ is a function from $\mathbb{R}$ to $\mathbb{R}$. For which values of a and b does f have an attracting fixed point? A repelling fixed point?
4. Let $f(x) = x + x^3$. Find all fixed points of f and decide whether they are attracting or repelling.
5. Give an example of a function $h(x)$ for which $h'(0) = 1$ and $x = 0$ is an attracting fixed point.
6. Draw a transition graph for a tripling map $T(x) = 3x(\text{mod } 1)$ and prove from this graph that there are points of arbitrary periods.
7. Prove that the tripling map has sensitive dependence on initial conditions on $[0,1]$.
8. Let $G(x) = 4x(1-x)$ be the logistic map. Prove that there are points in $[0,1]$ that are not fixed points, periodic points, or eventually periodic points of G.
9. Define $x_{n+1} = \frac{x_n+2}{x_n+1}$.
   (a) For $x_0 > 0$ find $L = \lim_{n\to\infty} x_n$.
   (b) What negative points $x_0 < 0$ lie in the basin of attraction of L?
10. For the map $g(x) = 3.05x(1-x)$ find the stability of all fixed points and period-2 points.
11. Let $f : [0,\infty) \to [0,\infty)$ be a smooth (continuously differentiable) function, $f(0) = 0$, and let $p > 0$ be a fixed point such that $f'(p) \geq 0$. Assume further that $f'(x)$ is decreasing on $[0,\infty)$. Prove that all positive x converge to p under iterations of f.
12. Write down a ternary expansion of $\frac{3}{5}$.
13. For a tripling map T find a point x such that the orbit of x under T is dense in $[0,1]$.
14. For a tripling map T find a point x which is not periodic and whose orbit under T is not dense in $[0,1]$.
15. How many attracting orbits can a function $f(x) = ax^2 + bx + c$ have?
16. Which of the following sets are dense in $[0,1]$?
   (a) the set of all points in $[0,1]$ whose ternary expansions contain each digit infinitely many times;

(b) the set of all points in $[0, 1]$ whose binary expansions contain some digit only finitely many times;

(c) the set of all points in $[0, 1]$ whose ternary expansions do not contain digit 1;

(d) $[0, 1] - X$, where X is arbitrary finite set;

(e) the set of rational multiples of π.

17. Prove that a point $\frac{1}{4}$ belongs to the middle third Cantor set.

18. Is the tent map $T(x)$ expansive? Why?

19. For which values of a and b does the map $f(x) = ax + b$ have a sensible dependence on initial conditions?

20. Prove that for any polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree at least 2 the Julia set of f is bounded.

21. Does the orbit of the point $i + 1$ stay bounded under iterations of $f(z) = z^2 - 1$?