Math 372 (Dynamical Systems), Section 1 Your name: KEY

Quiz 9

November 19, 2010

1. Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3 - \frac{x^2}{4}$. Prove that f has exactly one attracting periodic orbit. Find this orbit. Hint: recall that the Schwarzian derivative is defined as $S_f(x) = \frac{f'''(x)f'(x) - \frac{3}{2}f''(x)^2}{f'(x)^2}$.

We have $f'(x) = -\frac{x}{2}$, $f''(x) = -\frac{1}{2}$, $f'''(x) = 0$, so

$$S_f(x) = \frac{0 - \frac{3}{2}\cdot\frac{1}{4}}{\frac{x^2}{4}} = -\frac{3}{2x^2} < 0 \text{ whenever } f'(x) \neq 0.$$

So each attracting periodic orbit must contain the critical point in the basin, and we can't have more than 1 attracting periodic orbit.

Fixed pts: $f(x) = x \Rightarrow x = -6$ or $x = 2$.

$|f'(-6)| = 3 \Rightarrow -6$ is repelling

$|f'(2)| = 1$, so we need more.

We have $(f^2(x))'' = -\frac{3}{16}x^2 + \frac{3}{4} = \begin{cases} <0, & x>2, \\ =0, & x=2, \\ >0, & x<2, \end{cases}$ so $f^2(x) = \begin{cases} >x, & x<2, \\ =x, & x=2, \\ <x, & x>2. \end{cases}$

Therefore $x=2$ is the only attracting fixed point (and orbit)

2. Prove that the maps $f, g: [0,\infty) \to [0,\infty)$ defined by $f(x) = \frac{x}{2}$ and $g(y) = \frac{y}{3}$ are topologically conjugate. Hint: look for a conjugator of the form $h(x) = x^\alpha$ for $\alpha \in \mathbb{R}$.

If h conjugates f to g, then

$$h(f(x)) = g(h(x))$$

$$\begin{tikzcd} {[0,\infty)} \arrow[r, "f"] \arrow[d, "h"] & {[0,\infty)} \arrow[d, "h"] \\ {[0,\infty)} \arrow[r, "g"] & {[0,\infty)} \end{tikzcd}$$

If $h(x) = x^\alpha$, then

$$\left(\frac{x}{2}\right)^\alpha = \frac{x^\alpha}{3}$$

$$\frac{x^\alpha}{2^\alpha} = \frac{x^\alpha}{3} \Rightarrow 2^\alpha = 3 \Rightarrow \alpha = \log_2 3.$$

Therefore $h(x) = x^{\log_2 3}$ conjugates f to g.