Math 372 (Dynamical Systems), Section 1 Your name: KEY

Quiz 6

October 20, 2010

1. Consider a system of differential equation

$$\dot{x} = y,$$
$$\dot{y} = -x - y.$$

- Find all fixed points of this system.

$$\left.\begin{aligned} \dot{x} &= y = 0 \\ \dot{y} &= -x - y = 0 \end{aligned}\right\} \Rightarrow (0,0) \text{ is the only fixed point}$$

- Find a weak Lyapunov function for this system and check that it is nonincreasing along the trajectories.

$$L(x,y) = \frac{y^2}{2} - \int_0^x (-s)\,ds = \frac{x^2}{2} + \frac{y^2}{2}$$

$$\frac{d}{dt}L(x(t),y(t)) = x \cdot \dot{x} + y \cdot \dot{y} = xy - y(x+y) = -y^2 \leq 0$$

Also for each $(x,y) \in \mathbb{R}^2$: $L(x,y) = \frac{x^2}{2} + \frac{y^2}{2} \geqslant 0 = L(0,0)$.

Thus L is a weak Lyapunov function.

- For each attracting fixed point determine its basin of attraction. Indicate the method that you use and check the requirements of the method.

Method I: we have a system of linear equations with matrix $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, for which

$$\Delta = 1 > 0$$
$$\tau = -1 < 0$$
$$\tau^2 - 4\Delta = -3 < 0,$$

so we have a stable focus and the basin of attraction of $(0,0)$ is $\mathbb{R}^2$

Method II: For any $C > 0$ the set

$U_C = \{(x,y) : L(x,y) \leq C\} = \{(x,y) : \frac{x^2}{2} + \frac{y^2}{2} \leq C\}$ is in the basin of attraction of $(0,0)$ because $\frac{d}{dt}L(x,y) < 0$ unless $y = 0$ and if $y = 0$, $\dot{y} = -x \neq 0$ if $x \neq 0$, so $\varphi(t, (x, 0))$ leaves line $y = 0$ when $t > 0$. Thus $\mathbb{R}^2 = \bigcup_{C>0} U_C$ is the basin of attraction of $\bar{0}$.