Math 372 (Dynamical Systems), Section 1 Your name: KEY

Quiz 2

September 21, 2010

1. Consider the system of linear differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where the matrix

$$A = \begin{bmatrix} 5 & 0 & 6 \\ -1 & 2 & -1 \\ -3 & 0 & -4 \end{bmatrix},$$

which has eigenvalues -1, 2 and 2. Find the general solution.

First we find eigenvectors.

$$\lambda = -1: \quad A + I = \begin{bmatrix} 6 & 0 & 6 \\ -1 & 3 & -1 \\ -3 & 0 & -3 \end{bmatrix} \xrightarrow[r_3 \to r_3 + 3r_1]{r_1 \to r_1/6} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2 \to r_2 + r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\bar{v}^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$

$$\lambda = 2: \quad A - 2I = \begin{bmatrix} 3 & 0 & 6 \\ -1 & 0 & -1 \\ -3 & 0 & -6 \end{bmatrix} \xrightarrow[r_1 \to r_1/3]{r_3 \to r_3 + r_1} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2 \to r_2 + r_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\bar{v}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the only eigenvector for $\lambda = 2$.

Need to find a generalized eigenvector $\bar{v}^3$ as a solution to $(A - 2I)\bar{v}^3 = \bar{v}^2$.

$$\left[\begin{array}{ccc|c} 3 & 0 & 6 & 0 \\ -1 & 0 & -1 & 1 \\ -3 & 0 & -6 & 0 \end{array}\right] \xrightarrow[r_1 \to r_1/3]{r_3 \to r_3 + r_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \to r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \to r_1 - 2r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $\bar{v}^3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is a generalized eigenvector for $\lambda = 2$

Thus, the general solution is

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) =$$

$$= c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}.$$