1. (20 points) For the following linear system of differential equations

$$\begin{cases} \dot{x} = -3x + y, \\ \dot{y} = -2x. \end{cases}$$

(*a*) solve for an explicit solution

The matrix corresponding to the system is $\begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$. The eigenvalues: $\chi(\lambda) = (-\lambda-3)(-\lambda)+2 = \lambda^2+3\lambda+2 = (\lambda+1)(\lambda+2)$ $\lambda=-1$ $\lambda=-2$

Eigenvector for $\lambda=-1$:

$$\begin{bmatrix} -3+1 & 1 \\ -2 & 0+1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow \bar{v}^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvector for $\lambda=-2$

$$\begin{bmatrix} -3+2 & 1 \\ -2 & +2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \Rightarrow \bar{v}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

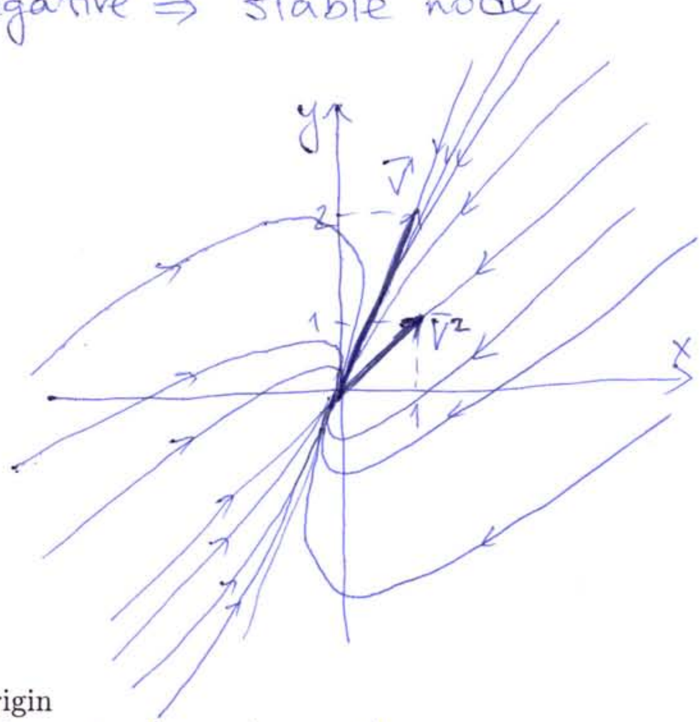
General solution

$$\bar{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(*b*) determine the type of the system (saddle, focus, node, etc.)

Both eigenvalues are negative $\Rightarrow$ stable node

(*c*) sketch a phase portrait



(*d*) find the stable manifold of the origin

Every point is attracted to the origin, so $W^s(\bar{0}) = \mathbb{R}^2$.

2. (20 points)

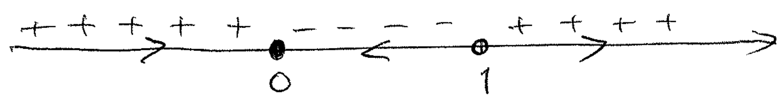
For the following differential equation

$$\dot{x} = x^2 - x$$

(*a*) find all fixed points

$\dot{x} = x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow \begin{matrix} x=0 \\ x=1 \end{matrix}$ are fixed points

(*b*) plot a phase portrait



(*c*) solve for an explicit solution $\varphi(t, x_0)$ satisfying $\varphi(0, x_0) = x_0$

$$\frac{dx}{x(x-1)} = dt$$

$$\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = \int dt = t + C$$

$$\ln\left|\frac{x-1}{x}\right| = t + C$$

$$\frac{x-1}{x} = ce^t$$

$$x = \frac{1}{1 - ce^t}$$

Plug $x(0) = x_0$ to get C:

$$x_0 = \frac{1}{1-C} \Rightarrow C = 1 - \frac{1}{x_0} = \frac{x_0 - 1}{x_0}.$$

Therefore

$$\varphi(t, x_0) = \frac{1}{1 - \frac{x_0 - 1}{x_0} e^t} = \frac{x_0}{x_0 - (x_0 - 1)e^t}$$

(*d*) what is the maximal interval where the flow $\phi(t, \mathbf{x}_0)$ is defined for $x_0 = 2$?

When $x_0 = 2$ we have

$\varphi(t, 2) = \dfrac{2}{2 - e^t}$. It is undefined when $2 - e^t = 0$, or $e^t = 2$, so $t = \ln 2 > 0$.

Therefore, maximal interval where the solution $\varphi(t, 2)$ is defined is $(-\infty, \ln 2)$.

3. (20 points) For the following system of differential equations

$$\begin{cases} \dot{x} = y - x^2, \\ \dot{y} = x - y. \end{cases}$$

(*a*) find all fixed points

$$\left.\begin{matrix} \dot{x} = y - x^2 = 0 \\ \dot{y} = x - y = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x = y \\ y - y^2 = 0 \end{matrix}\right\} \Rightarrow \begin{bmatrix} y = 0 \\ x = 0 \end{bmatrix} \text{ or } \begin{bmatrix} y = 1 \\ x = 1 \end{bmatrix}.$$

So $(0,0)$ and $(1,1)$ are all fixed points.

(*b*) for each fixed point determine the linearized system at this point and classify its type as saddle, focus, node, etc.

$$DF_{(x,y)} = \begin{bmatrix} -2x & 1 \\ 1 & -1 \end{bmatrix}. \quad \text{So}$$

$@(0,0)$: $DF_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. $\Delta = 0 \cdot (-1) - 1 \cdot 1 = -1 < 0 \Rightarrow (0,0)$ is a saddle

$@(1,1)$: $DF_{(1,1)} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

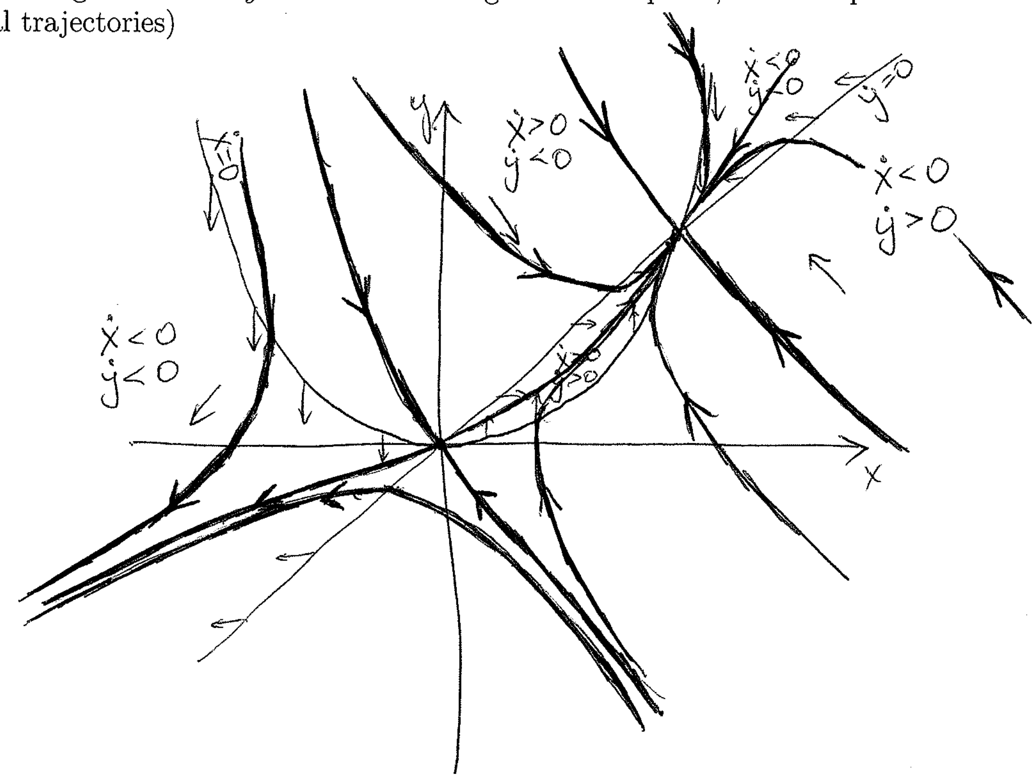
$$\left.\begin{matrix} \Delta = (-2)(-1) - 1 \cdot 1 = 1 > 0 \\ \tau = -2 - 1 = -3 < 0 \\ \tau^2 - 4\Delta = 9 - 4 = 5 > 0 \end{matrix}\right\} (1,1) \text{ is a stable node}$$

(*c*) determine the nullclines and signs of $\dot{x}$ and $\dot{y}$ in the various regions of the plane; sketch a phase portrait (including typical trajectories)

Nullclines:

$\dot{x} = 0 \Rightarrow y = x^2$

$\dot{y} = 0 \Rightarrow y = x$



4. (20 points) Consider the following system of differential equations

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \frac{1}{2}. \end{cases}$$

(*a*) compute the total energy function

$$E(x,y) = \frac{y^2}{2} + V(x) = \frac{y^2}{2} - \int_0^x \frac{ds}{2} = \frac{y^2}{2} - \frac{x}{2}$$

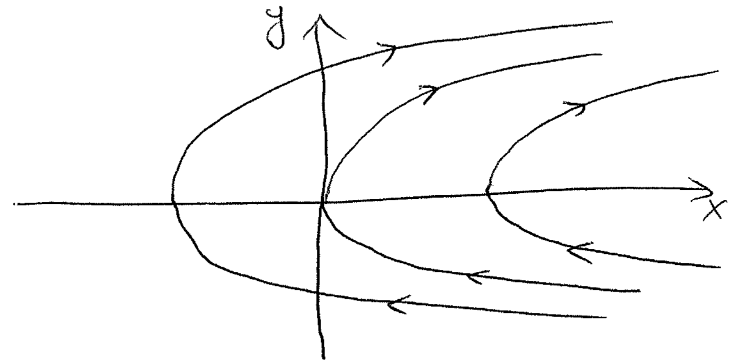
(*b*) using the results in (*a*) sketch a phase portrait

Level curves $E^{-1}(C)$ have equations:

$$\frac{y^2}{2} - \frac{x}{2} = C,$$

so

$x = y^2 - 2C$ - parabolas



5. (20 points) For the following system of differential equations

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^3 - y^3, \end{cases}$$

the only fixed point is $(0,0)$.
(*a*) find a Lyapunov function $L(x,y)$

Let $L(x,y) = \frac{y^2}{2} - \int_0^x (-s^3)\,ds = \frac{y^2}{2} + \frac{x^4}{4}$

(*b*) check that $(0,0)$ is a point where L attains its global minimum

For each $(x,y) \in \mathbb{R}^2$:

$L(x,y) = \frac{y^2}{2} + \frac{x^4}{4} \geqslant \frac{0^2}{2} + \frac{0^4}{4} = L(0,0)$

(*c*) check that L is nonincreasing along the solutions of a system (checking these conditions will guarantee that the basin of attraction of $(0,0)$ is $\mathbb{R}^2$).

$$\frac{d}{dt} L(x(t), y(t)) = y \cdot \dot{y} + x^3 \dot{x} = y(-x^3 - y^3) + x^3 \cdot y = -y^4 \leq 0.$$

Thus L is nonincreasing along the trajectories.