Solution to problem 5.3.2.

We have a system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x + L, \end{cases}$$

where $0 < L < 1$.

The fixed points of the system are those that satisfy $y = 0$ and $\sin x = L$, so these are

$$\mathbf{x}_n^* = (\arcsin L + 2\pi n, 0)$$

and

$$\mathbf{y}_n^* = (\pi - \arcsin L + 2\pi n, 0).$$

The types of the fixed points can be determined by analysis of the linearized systems.

We have

$$DF_{(x,y)} = \begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}.$$

So for fixed points $\mathbf{y}_n^*$ we have

$$DF_{\mathbf{y}_n^*} = \begin{bmatrix} 0 & 1 \\ -\cos(\pi - \arcsin L + 2\pi n) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sqrt{1-L^2} & 0 \end{bmatrix}.$$

For this matrix we have $\Delta = -\sqrt{1-L^2} < 0$, so the fixed points $\mathbf{y}_n^*$ are saddles.

However, for the fixed points $\mathbf{x}_n^*$ the matrix of linearized system has form:

$$DF_{\mathbf{x}_n^*} = \begin{bmatrix} 0 & 1 \\ -\cos(\arcsin L + 2\pi n) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sqrt{1-L^2} & 0 \end{bmatrix},$$

for which we have $\Delta = \sqrt{1-L^2}$ and $\tau = 0$, so these fixed points are linear centers and $\mathbf{x}_n^*$ are not hyperbolic.

Hence, in order to describe stability of these points (for nonlinear system) we need more accurate analysis. We will analyze the energy function.

The potential energy is

$$V(x) = -\int_0^x (-\sin s + L)\, ds = -\cos x + 1 - Lx$$

and the total energy is

$$E(x,y) = \frac{y^2}{2} - \cos x + 1 - Lx.$$

The graph of $V(x)$ is depicted in Figure 1 and the graph of potential is shown in Figure 2.

The critical points of the potential coincide with x-coordinates of fixed points of the system since

$$\dot{V}(x) = \frac{d}{dx}\left(-\int_0^x (-\sin s + L)\, ds\right) = \sin x - L.$$

For points $\mathbf{x}_n^*$ the matrix of second derivatives of $E(x,y)$ is

$$\begin{bmatrix} \frac{\partial^2 E}{\partial x^2} & \frac{\partial^2 E}{\partial x \partial y} \\ \frac{\partial^2 E}{\partial x \partial y} & \frac{\partial^2 E}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \cos(\arcsin L + 2\pi n) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{1-L^2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Since both upper left entry and the determinant of this matrix are positive, points $\mathbf{x}_n^*$ are points at which $E(x,y)$ attains local minima. Therefore the level sets of $E(x,y)$ will be closed curves in the neighborhoods of these fixed points, which shows that $\mathbf{x}_n^*$ are nonlinear centers surrounded by periodic orbits. The phase portrait is sketched in Figure 3.

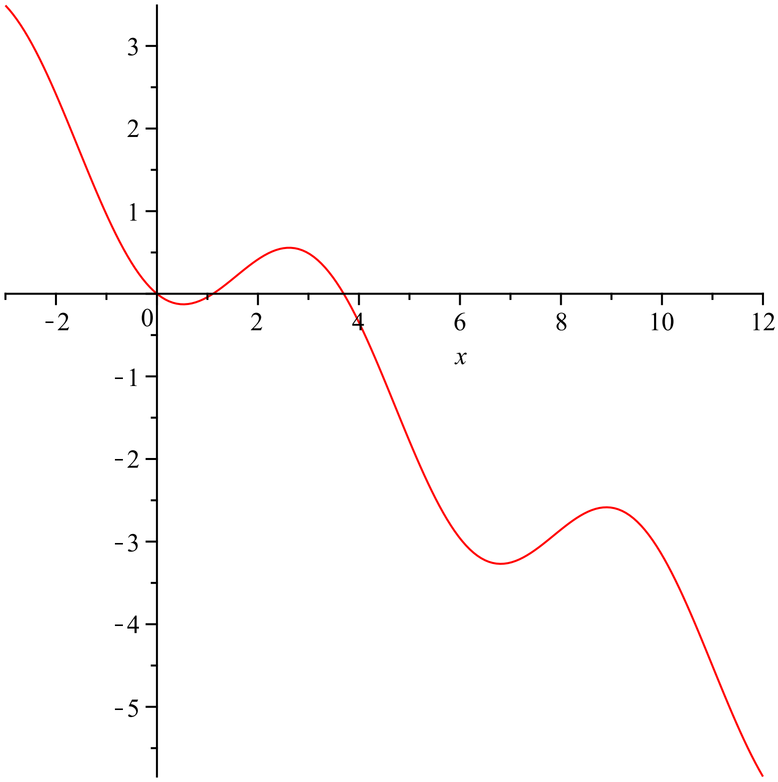

Figure 1: Potential function

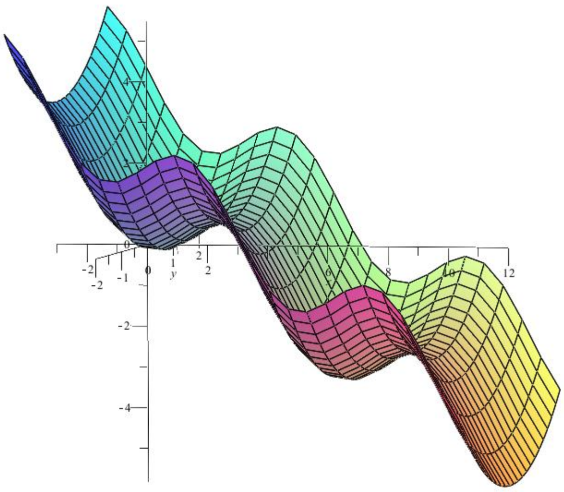

Figure 2: Total energy

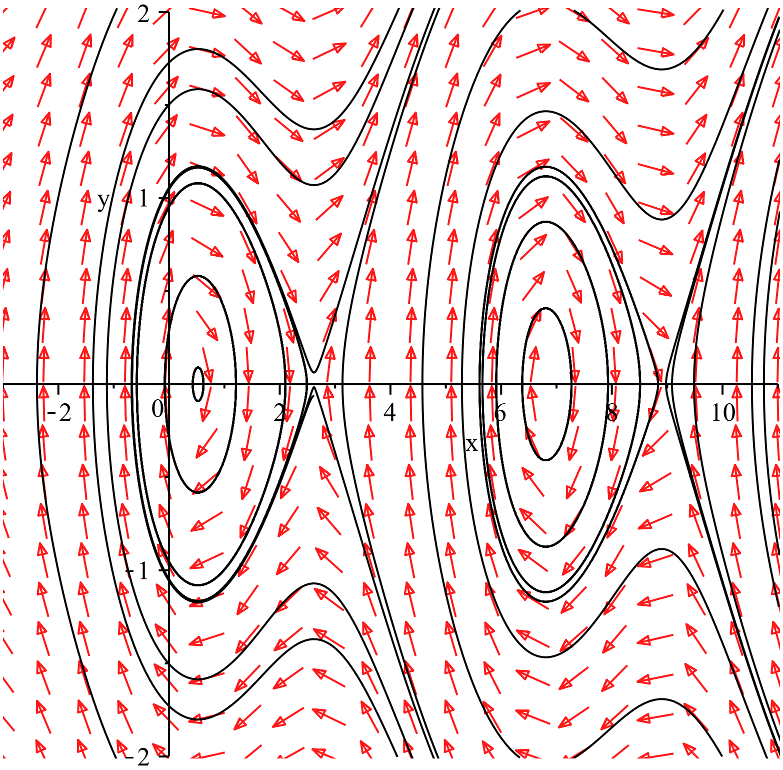

Figure 3: Phase portrait