Math 304 (Linear Algebra), Section 2 Your name: KEY

Quiz 11

December 3, 2010

Consider the matrix

$$A = \begin{bmatrix} 1 & 9 \\ +1 & 1 \end{bmatrix}$$

1. Find all eigenvalues of A.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 9 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 9 = (1-\lambda-3)(1-\lambda+3) =$$

$$= (\lambda - 4)(\lambda + 2).$$ So eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -2$.

2. Find eigenvectors for all eigenvalues of A.

$\lambda_1 = 4$: $A - 4I = \begin{bmatrix} 1-4 & 9 \\ 1 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \xrightarrow[r_2 \to r_2 + 3r_1]{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = 3x_2$, $x_2 = x_2$

So $\bar{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 4$

$\lambda_2 = -2$: $A + 2I = \begin{bmatrix} 1+2 & 9 \\ 1 & 1+2 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix} \xrightarrow[r_2 \to r_2 - 3r_1]{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = -3x_2$, $x_2 = x_2$

So $\bar{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -2$

3. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Let $X = (\bar{v}_1, \bar{v}_2)$ be the basis consisting of eigenvectors. Then ${}_X A_X = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and

$A = ({}_E I_X) \cdot {}_X A_X \cdot ({}_E I_X)^{-1}$, where ${}_E I_X = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$.

So one can take $P = {}_E I_X = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$ and

$D = {}_X A_X = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$