

## Quiz 4

March 1, 2010

1. The sequence  $(x_n)_{n \in \mathbb{N}}$  is defined recursively by

$$x_1 = 1, x_2 = 2, \quad x_{n+2} = x_{n+1} + x_n - 1$$

(a) Find  $x_8$ .

$$\begin{aligned} x_3 &= 2+1-1=2, \quad x_4 = 2+2-1=3, \\ x_5 &= 3+2-1=4, \quad x_6 = 4+3-1=6 \\ x_7 &= 6+4-1=9, \quad x_8 = 9+6-1=14 \end{aligned}$$

- (b) Prove that  $x_n \leq 2^n$  for all  $n \in \mathbb{N}$ .

Use induction.

1) Let  $P(n) = [x_n \leq 2^n]$

2) Induction base:

$$P(1) = [x_1 \leq 2^1] = [1 \leq 2] \text{ is true}$$

3) Induction step:

Assume  $P(k)$  is true for all  $k$  s.t.  ~~$1 \leq k \leq n$~~ .

If  $n=1$ , then  $\overset{P(n+1)}{P(2)} = [2 \leq 2^2]$  is true, so we can assume  $n \geq 2$ . In this case

$$x_{n+1} = x_n + x_{n-1} - 1 \leq 2^n + 2^{n-1} - 1 \leq 2^n + 2^n = 2^{n+1}.$$

↑  
induction  
assumption

Thus  $P(n+1)$  is true.

Therefore by mathematical induction,  
 $P(n)$  is true for all  $n \in \mathbb{Z}$ .