Math 330: Number Systems, Section 3, Spring 2010 — Project Assignment
First submission is due April 26, second submission is due May 5

In this project you will create rational numbers step-by-step from integers and study their properties. Be sure to include all details in your proofs.
On the set $\mathbb{Z}\times(\mathbb{Z}-\{0\})$ define a relation $\sim$ by declaring $(a,b)\sim(c,d)$ if and only if $ad=bc$.

Problem 1.

(1) Give an example of two different elements of $\mathbb{Z}\times(\mathbb{Z}-\{0\})$ which are equivalent with respect to $\sim$.
(2) Prove that $\sim$ is an equivalence relation.

For all $(a,b),(c,d)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$ define $(a,b)+(c,d)=(ad+bc,bd)$ and $(a,b)\cdot(c,d)=(ac,bd)$.

Problem 2. *Prove that if $(a_1,b_1)\sim(a_2,b_2)$ and $(c_1,d_1)\sim(c_2,d_2)$, then $(a_1,b_1)+(c_1,d_1)\sim(a_2,b_2)+(c_2,d_2)$ and $(a_1,b_1)\cdot(c_1,d_1)\sim(a_2,b_2)\cdot(c_2,d_2)$.*

Let $[a,b]$ denote the equivalence class with respect to $\sim$ of $(a,b)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$, and define $\mathbb{Q}$ to be the set of equivalence classes of $\mathbb{Z}\times(\mathbb{Z}-\{0\})$. The set $\mathbb{Q}$ is called the *set of rational numbers*. An equivalence class $[a,b]$ is called a *rational number*. Note that the more common notation for $[a,b]$ is $\frac{a}{b}$, but we will use the notation $[a,b]$ to emphasize the way we created rational numbers.

For all $[a,b],[c,d]\in\mathbb{Q}$ define $[a,b]+[c,d]=[(a,b)+(c,d)]$ and $[a,b]\cdot[c,d]=[(a,b)\cdot(c,d)]$; these definitions make sense, i.e., they do not depend on the choice of representatives, because of problem 2.

The following properties hold for all $[a,b],[c,d],[e,f]\in\mathbb{Q}$:

(i) $([a,b]+[c,d])+[e,f]=[a,b]+([c,d]+[e,f])$;
(ii) $[a,b]+[c,d]=[c,d]+[a,b]$;
(iii) $([a,b]\cdot[c,d])\cdot[e,f]=[a,b]\cdot([c,d]\cdot[e,f])$;
(iv) $[a,b]\cdot[c,d]=[c,d]\cdot[a,b]$;
(v) $([a,b]+[c,d])\cdot[e,f]=([a,b]\cdot[e,f])+([c,d]\cdot[e,f])$;
(vi) there exists $\mathbf{0}\in\mathbb{Q}$ such that for all $[a,b]\in\mathbb{Q}$, $\mathbf{0}+[a,b]=[a,b]$;
(vii) for every $[a,b]\in\mathbb{Q}$ there exists $-[a,b]\in\mathbb{Q}$ such that $-[a,b]+[a,b]=\mathbf{0}$;
(viii) there exists $\mathbf{1}\in\mathbb{Q}-\{\mathbf{0}\}$ such that for all $[a,b]\in\mathbb{Q}$, $\mathbf{1}\cdot[a,b]=[a,b]$;
(ix) for every $[a,b]\in\mathbb{Q}-\{\mathbf{0}\}$ there exists $[a,b]^{-1}\in\mathbb{Q}$ such that $[a,b]^{-1}\cdot[a,b]=\mathbf{1}$.

Problem 3. *Prove properties* (vi), (vii), (viii), *and* (ix) *(notice that in particular you need to explicitly define $\mathbf{0},-[a,b],\mathbf{1},[a,b]^{-1}$). Prove also at least one of the remaining properties.*

Problem 4. *Among the properties* (i)–(ix) *above, which ones hold and which ones fail in $\mathbb{Z}\times(\mathbb{Z}-\{0\})$? Justify your answer.*

For $[a,b],[c,d]\in\mathbb{Q}$ define $[a,b]\le[c,d]$ if and only if ($bd>0$ and $ad\le bc$) or ($bd<0$ and $ad\ge cb$).

Problem 5. *Prove that the above definition of $\le$ does not depend on the choice of representatives.*

Problem 6. *Prove that the following properties hold for all $[a,b],[c,d],[e,f]\in\mathbb{Q}$:*
(a) $[a,b]\le[c,d]$ or $[c,d]\le[a,b]$;
(b) if $[a,b]\le[c,d]$ and $[c,d]\le[a,b]$, then $[a,b]=[c,d]$;
(c) if $[a,b]\le[c,d]$ and $[c,d]\le[e,f]$, then $[a,b]\le[e,f]$.

In the last problem you will show how to embed integers in rationals.

Problem 7. *Define (find) a function $f\colon\mathbb{Z}\to\mathbb{Q}$ satisfying the following properties:*
(A) $f(0)=\mathbf{0}$ and $f(1)=\mathbf{1}$;
(B) for all $m,n\in\mathbb{Z}$, $f(m+n)=f(m)+f(n)$, $f(m\cdot n)=f(m)\cdot f(n)$, and if $m\le n$ then $f(m)\le f(n)$;
(C) for all $[a,b]\in\mathbb{Q}$ there exists $n\in\mathbb{N}$ such that $[a,b]\le f(n)$.
Prove also that f is injective but not surjective.