Homework assignment 14

This is the second part of a preparational homework for the final. I will not collect this homework, but I encourage you to solve these problems before final.

1. Prove that for each $n \in \mathbb{N}$ we have that $6^n + 7^{2n+3}$ is divisible by 43.

2. Negate the following statements

   (a) If every animal has to breath, birds have to breath too.

   (b) For all real numbers x and y such that $x < y$ there exists a rational number z such that $x < z < y$.

   (c) Every subset of countable set is countable.

   (d) There is a surjection from A to B if and only if there is an injection from B to A.

   (e) Every planet in the Solar system is either black or white.

3. Give the following definitions using statements with quantifiers

   (a) A number $n \in \mathbb{N}$ is prime.

   (b) A number $n \in \mathbb{N}$ is composite.

   (c) A sequence $(a_n)_{n\in\mathbb{N}}$ is convergent.

   (d) A sequence $(a_n)_{n\in\mathbb{N}}$ is divergent.

   (e) A sequence $(a_n)_{n\in\mathbb{N}}$ is bounded above.

   (f) A sequence $(a_n)_{n\in\mathbb{N}}$ is not bounded above.

   (g) A sequence $(a_n)_{n\in\mathbb{N}}$ is increasing.

   (h) A sequence $(a_n)_{n\in\mathbb{N}}$ is not increasing.

4. Let A, B, $C(x)$, $D(x,y)$ be statements. Negate the following statements.

   (a) $B \Rightarrow (A \vee C(x))$

   (b) $(A \Rightarrow B) \Leftrightarrow [\forall x(\overline{C(x)} \wedge (\exists y : D(x,y)))]$

   (c) $\forall x \exists y \forall z (A \vee B) \wedge \overline{D(x,y)} \wedge (z^2 < 1)$

5. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

   (a) Prove that $(x_n)_{n\in\mathbb{N}}$ is increasing.

   (b) Prove that for each $n \in \mathbb{N}$

$$x_n \leq 2 - \frac{1}{n}.$$

   (c) Conclude that $(x_n)_{n\in\mathbb{N}}$ is convergent and

$$\sum_{k=1}^{\infty} \frac{1}{k^2} := \lim_{n\to\infty} x_n \leq 2.$$

6. Let $f : A \to B$ be a function and let $U, V \subset A$ be two arbitrary subsets. Prove that $f(U \cup V) = f(U) \cup f(V)$.

7. Prove that if sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent, then sequence $(a_n - b_n)_{n\in\mathbb{N}}$ is also convergent and $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}(a_n) - \lim_{n\to\infty}(b_n)$.

8. Let A and B be bounded above subset of $\mathbb{R}$. Prove that
$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

9. (a) How many elements does $P(P(n))$ have?
   (b) Describe all elements of $P(P(P(\emptyset)))$.

10. Let A be a set consisting of pairwise disjoint intervals of real numbers. Prove that A is countable (finite or countably infinite).

11. Assume $\Diamond \notin \mathbb{N}$. Prove that $\mathbb{N}$ and $\mathbb{N} \cup \{\Diamond\}$ have the same cardinality.

12. Prove that for any $k \in \mathbb{N}$ sets $\mathbb{N}$ and $\mathbb{N} - \{k\}$ have the same cardinality.

13. Prove that for any finite subset A of $\mathbb{N}$ sets $\mathbb{N}$ and $\mathbb{N} - A$ have the same cardinality.

14. Prove that any subset of a countable set is countable.