Math 371 (Differential Equations), Section 3 (Dmytro Savchuk) Fall 2009

Homework assignment 8 (with answers)

Note: I will not collect this assignment – just do it for your benefit. This is, perhaps, the most important HW assignment in the course. This is a preparational homework for the final that covers the topics that will be presented on the final. Here is a list of these topics:

1. First order equations using exactness or integrating factors
2. Euler-Cauchy problem
3. Orthogonal family of curves
4. 2×2 systems of linear equations with distinct roots for the polynomial
5. Power series with solution to 5th order
6. Second solution using reduction of order
7. Laplace

Solve the following problems

1. Check that the following differential equation is exact and solve it

$$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0$$

Answer: $y = \dfrac{C - x}{2 + xe^x}$

2. Make the following equation exact by multiplying by appropriate integrating factor and solve it

$$(2x^2 + y)dx + (x^2 y - x)dy = 0$$

Answer: $2x - yx^{-1} - \dfrac{y^2}{2} = C$

3. Solve the Euler-Cauchy equation

$$x^2 y'' - 5xy' + 8y = 0$$

Answer: $y\,(x) = C_1 x^2 + C_2 x^4$

4. Solve the Euler-Cauchy equation

$$x^2 y'' - 5xy' + 9y = 0$$

Answer: $y\,(x) = C_1 x^3 + C_2 x^3 \ln x$

5. Solve the Euler-Cauchy equation

$$x^2 y'' - 5xy' + 10y = 0$$

Answer: $y\,(x) = C_1 x^3 \sin(\ln x) + C_2 x^3 \cos(\ln x)$

6. Find a family of curves orthogonal to the family $y = kx^4$.

   Answer: $x^2 + 4y^2 = C$

7. Find a family of curves orthogonal to the family $y^2 = kx$.

   Answer: $y^2 + 2x^2 = C$

8. Solve the system of differential equations

$$\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$$

   Answer:

$$\begin{cases} y_1 = 2C_1 + C_2 e^{3t} \\ y_2 = -C_1 + C_2 e^{3t} \end{cases}$$

9. Find the solution of differential equation $2y'' + xy' + y = 0$ using power series. Approximate the solution by taking explicitly the first 6 terms in your power series (up to x^5).

   Answer:

$$y(x) = a_0 + a_1 x - \frac{a_0}{4}x^2 - \frac{a_1}{6}x^3 + \frac{a_0}{32}x^4 + \frac{a_1}{60}x^5 + \ldots$$

10. Find the solution of differential equation $x^2y'' + 3y' - xy = 0$ using power series. Approximate the solution by taking explicitly the first 6 terms in your power series (up to x^5).

    Answer: The standard method allows to find just one (linearly independent) solution:

$$y(x) = a_0\left(1 + \frac{1}{6}x^2 - \frac{1}{27}x^3 + \frac{7}{216}x^4 - \frac{23}{810}x^5 + \ldots\right)$$

11. Given that $f(x) = e^x$ is the solution to the differential equation $xy'' - (x+1)y' + y = 0$ find the second linearly independent solution.

    Answer: $y_2(x) = -x - 1$ (don't forget to divide by x to convert the equation into the standard form).

12. Find the Laplace transforms of the functions

    (a) $f(t) = e^{-2t}\sin 2t + e^{3t}t^2$

    Answer: $F(s) = \dfrac{2}{(s+2)^2+4} + \dfrac{2}{(s-3)^3}$

    (b) $f(t) = sin^2(4t)$

    Answer: $F(s) = \dfrac{32}{s(s^2+64)}$

    (c) $f(t) = t^2u(t-4)$

    Answer: $F(s) = e^{-4s}\left(\dfrac{2}{s^3} + \dfrac{8}{s^2} + \dfrac{16}{s}\right)$

    (d) $f(t) = \begin{cases} t^2, & t < 2, \\ \sin(t), & 2 < t < 4, \\ e^{-t}, & t > 4. \end{cases}$

    Hint: Write $f(t) = t^2u(t-2) + \sin t\big(u(t-2) - u(t-4)\big) + e^{-t}u(t-4)$. Then use second shifting theorem and the fact that $\sin(t+a) = \sin t\cos a + \cos t \sin b$.

    Answer:

$$F(s) = \frac{e^{-4-4\,s}}{s+1} + \frac{e^{-4\,s}\left(-\sin(4)\,s - \cos(4)\right) + e^{-2\,s}\left(\cos(2) + \sin(2)\,s\right)}{s^2+1} + 2\,\frac{e^{-2\,s}\left(2\,s^2 + 2\,s + 1\right)}{s^3}$$

13. Find the inverse Laplace transforms of the functions

(a) $F(s) = e^{-s}\dfrac{s}{s^2+4}$

Answer: $f(t) = \cos\big(2(t-1)\big)u(t-1)$

(b) $F(s) = \dfrac{s+1}{s^2+2s+10}$

Answer: $f(t) = e^{-t}\cos(3t)$

(c) $F(s) = \dfrac{5s^2+34s+53}{(s+3)^2(s+1)}$

Answer: $f(t) = (2\,t-1)\,e^{-3\,t} + 6\,e^{-t}$

(d) $F(s) = \ln\dfrac{s+2}{s-5}$

Answer: $f(t) = \dfrac{e^{5t}}{t} - \dfrac{e^{-2t}}{t}$

14. Use Laplace transform to solve the initial value problem

$$y' - y = u(t-1), \qquad y(0) = 0$$

Answer: $y(t) = (e^{t-1} - 1)u(t-1)$

15. Use the Laplace transform to solve the initial value problem

$$y' - y = u(t-1), \qquad y(0) = 0$$

Answer: Same as the previous problem (copy/paste issue...)

16. Use the Laplace transform to solve the initial value problem

$$y' - 2y = u(t-2), \qquad y(0) = 1$$

Answer: $y(t) = \dfrac{1}{2}\big(e^{2t-4} - 1\big)u(t-2) + e^{2t}$

17. Use the Laplace transform to solve the initial value problem

$$y'' + 9y = 3\delta(t-\pi), \qquad y(0) = 1, \quad y'(0) = 0$$

Answer: $y(t) = \cos(3t) + \sin\big(3(t-\pi)\big)u(t-\pi)$

18. Use the Laplace transform, the method of undetermined coefficients and the variation of parameters to solve initial value problem

$$y'' - 2y' + 5y = -8e^{-t}, \qquad y(0) = 2, \quad y'(0) = 12$$

Answer: $y(x) = C_1 e^x \sin(2\,x) + C_2 e^x \cos(2\,x) - e^{-x}$