Вісник Київського університету **2003, 1** *Bulletin of the University of Kiev*
Серія: фізико-математичні науки *Series:Physics & Mathematics*

УДК 512.544

Дмитро М. Савчук*

Проблема слів в групах автоморфізмів кореневих дерев зі стягуючою самоподібністю

Дана оцінка складності алгоритму, що розв'язує проблему слів в групах зі стягуючою самоподібністю.

Ключові слова: проблема слів, стягуюча самоподібність, ядро самоподібності

Dmytro M. Savchuk*

On word problem in contracting automorphism groups of rooted trees

An estimation of the complexity of an algorithm solving the word problem in contracting groups is given.

Key Words: word problem, contracting group, nucleus

*E-mail: dsavchuk@yahoo.com

1. Tree (i.e. graph without cycles) is called *rooted* if it has a fixed vertex called the *root* of the tree. A rooted tree is called *d-ary* if the degree of the root is equal to *d* and the degrees of all the other vertices are equal to *d*+1. We will consider in this paper the *d*-ary trees only. All *d*-ary rooted trees are isomorphic and we denote them by $T^{(d)}$.

An *automorphism* of the rooted tree $T^{(d)}$ is an automorphism of the tree fixing the root. The set of all automorphisms of the rooted tree $T^{(d)}$ form a group (with the operation of superposition), which is denoted by $\mathrm{Aut}\,T^{(d)}$.

The class of automorphism groups of *d*-ary trees has certain universal embedding properties and contains finitely generated groups with different extremal properties: Burnside groups ([1,2,3,4]), groups of intermediate growth ([5]), just infinite groups ([6]), etc. Because of this, these groups were investigated by many authors. During the investigation some different types of automorphisms and groups where marked out: finitary, finite-state, self-similar, contracting, branch, etc.

Self-similar automorphism groups of a *d*-ary tree have applications in symbolic dynamics, ergodic theory, theory of fractals and are studying intensively. These groups are also interesting from the algorithmic point of view. The class of self-similar groups contains another wide class: contracting groups. (All definitions will be given in Section 2).

The word problem in finite-state finitely generated groups is solvable (see [7]), but the algorithm has an exponential complexity. In this paper an estimation of the complexity of an algorithm solving the word problem in contracting groups is given. The following theorem is proved.

Theorem 1. *Let G be a contracting group which acts on the d-ary rooted tree and has an n-element self-similar generating set that contains the nucleus of the self-similar action. Then the word problem in this group is solvable and for* *any* $\varepsilon > 0$ *there exists an algorithm of polynomial complexity of degree* $(n^2 - 1)\log_2 d + \varepsilon$.

2. We will use a realization of the *d*-ary tree as the Hass diagrams of the set X^* of all finite words over the alphabet $X = \{x_1, x_2, \ldots, x_d\}$ with the prefix order. Then the empty word is the root of the tree and the vertices of the *n*th level are words of the length *n*. We well also consider the set $X^\omega = \{x_1 x_2 \ldots : x_i \in X\}$ of all right infinite words over the alphabet *X*.

Let $v = x_1 x_2 \ldots x_n \in X^*$ be a finite word. Let $T_v^{(d)}$ be the subtree with the root vertex *v* of the tree $T^{(d)}$ (the vertices of this subtree are the finite words of the type $\{vw : w \in X^*\}$). Let $g \in \mathrm{Aut}\, T^{(d)}$ be an arbitrary automorphism of the tree. We define the map $g|_v : X^* \to X^*$ by the rule $g|_v(x) = y \Leftrightarrow g(vx) = g(v)y$. It is a correctly defined automorphism of the tree $T_v^{(d)}$. But since the trees $T_v^{(d)}$ and $T^{(d)}$ are isomorphic, we can consider $g|_v$ as an automorphism of the tree $T^{(d)}$. The obtained automorphism is called the *restriction* of *g* in the word *v*.

Every automorphism *g* induces a permutation π on the set $X \subset X^*$ and *d* restrictions $g|_x$, $x \in X^1$. Moreover, every automorphism is uniquely determined by the permutation π and the function $x \mapsto g|_x$. Therefore it is possible to write every automorphism uniquely in the form

$$g = (g_1, \ldots, g_d)\pi \tag{1}$$

where $g_i \in \mathrm{Aut}\, T^{(d)}$ are the restrictions of *g* in the one-letter words and $\pi \in S_d$ is the action of *g* on *X*.

Let us write the multiplication rule for the automorphisms written in the form (1). Let $g, h \in \mathrm{Aut}\, T^{(d)}$ $g = (g_1, \ldots, g_d)\pi$, $h = (h_1, \ldots, h_d)\sigma$. Then we have:

$$g \cdot h = (g_1, \ldots, g_d)\pi \cdot (h_1, \ldots, h_d)\sigma = (g_1 h_{\pi(1)}, \ldots, g_d h_{\pi(d)})\pi\sigma. \tag{2}$$

This rule follows directly from the definition. It implies also the following formula for the inverse automorphism:

$$g^{-1} = \left((g_1, \ldots, g_d)\pi\right)^{-1} = (g^{-1}_{\pi^{-1}(1)}, \ldots, g^{-1}_{\pi^{-1}(d)})\pi^{-1} \tag{3}$$

Definition 1. A set $S \subset \mathrm{Aut}\, T^{(d)}$ is called *self-similar* if for every $g \in S$, and $x \in X$ there exist $h \in S$ and $y \in X$ such that for all $w \in X^\omega$ we have

$$g(xw)=yh(w).$$

In other words, the set *S* is self-similar if and only if all the restrictions of every element of *S* belong to *S*.

A group $G < \mathrm{Aut}\, T^{(d)}$ is called self-similar if it is self-similar as a set.

Self-similar groups are also often called *state-closed* or *semi-fractal*. We will consider the finitely generated self-similar groups only.

Definition 2. A group $G < \mathrm{Aut}\,T^{(d)}$ is said to be *finite-state* if for every $g \in G$ the set of the restrictions $\{g|_v : v \in X^*\}$ is finite.

We have the following proposition.

Proposition 1. A finitely generated self-similar group *G* has a finite self-similar generating set if and only if it is finite-state.

Proof. Let *S* be a self-similar finite generating set of the group *G*. Let $g \in G$ be an arbitrary element. It can be represented as a group word over *S* of some length *k*. But formulae (2), (3) and self-similarity of the generating set *S* imply that every restriction of *G* will be also represented as a group word over *S* of length not greater than *k*. Since the set of all the words of length *k* is finite, the set of restrictions of the element *G* is finite too. Thus, group *G* is finite-state.

Conversely, let the group *G* be finite-state and let *S* be its arbitrary finite generating set. Then we can get the desired self-similar generating set if we add to *S* the restrictions of all the elements of *S* in all the words in X^* (the set of such restriction is finite). □

An important notion is the notion of a contracting group.

Definition 3. A self-similar finitely generated group *G* is called *contracting* if there exists a finite self-similar set $S \subset G$ such that for each $g \in G$ and $x_1 x_2 x_3 \ldots \in X^{\omega}$ there exists $k \in \mathbf{N}$ that for all $n>k$ the restriction of *g* in the word $x_1 x_2 \ldots x_n$ belongs to *S*. Every such set *S* is called the *quasinucleus* of a contracting group *G*. The minimal quasinucleus is called the *nucleus* of the contracting group *G*.

We say, in conditions of the definition, that an element *g* contracts to *S* along the word $x_1 x_2 x_3 \ldots \in X^{\omega}$ at the *k*-th level. Note also that the identity automorphism always belongs to the nucleus.

Proposition 2. Each contracting group is finite-state.
The proof of this proposition is given in [8].

Corollary 1. Each contracting group has a self-similar finite generating set.

Proof of the last corollary follows immediately from Propositions 1 and 2. Note that construction of this self-similar finite generating set is constructive (provides that we know how the generators contract to the nucleus).

One of the most famous examples of self-similar groups are the Grigorchuk group and the “Adding machine”. The Grigorchuk group is the subgroup of $\mathrm{Aut}\,T^{(2)}$ generated by the automorphisms

$$\begin{cases} a = (1,1)\sigma, \\ b = (c, a), \\ c = (d, a), \\ d = (b,1), \end{cases}$$

where $\sigma = (1,2)$ is the transposition.

The Grigorchuk group has many interesting properties. For instance, it is an infinite finitely generated torsion group, it is a group of intermediate growth, it is just-infinite, has finite width, etc. This group is also contracting with the nucleus $\{1,a,b,c,d\}$.

Another example of a contracting group is the following. Let $a, b \in \mathrm{Aut}\, T^{(2)}$ be such that

$$\begin{cases} a = (1,1)\sigma, \\ b = (a, b)\sigma, \end{cases}$$

Then the group $G = < a, b >$ is contracting with the nucleus $\{1,a,b,b^{-1}\}$. It follows from the algorithm solving the word problem in contracting groups given in [8].

Let G be a finitely generated group and let S be its arbitrary finite generating set. We denote by $l(g)$ the word-length of $g \in G$ respectively to the generating set S (i.e., $l(g)$ is the minmal length of a word over S, representing g).

Definition 4. Let G be a finitely generated self-similar group. Then the limit

$$\rho = \overline{\lim_{k\to\infty}} \max_{v \in X^k} \sqrt[k]{\overline{\lim_{l(g)\to\infty}} \frac{l(g|_v)}{l(g)}}\,. \qquad (4)$$

is called *contracting coefficient* of the group G.

The following proposition shows that the contraction coefficient is well defined.

Proposition 3. Let G be a finitely generated self-similar group. Then the limit (4) exists and doesn't depend on the generating set of group G.

The proof of this proposition is given in [8].

Lemma 1. Let G be a finitely generated group with a contracting self-similar action. Let us take a number $M>0$ and a positive integer l_0 such that for every $g \in G$ and every word $v \in X^{l_0}$ of the length l_0 the inequality

$$l(g|_v) \le \frac{l(g)}{2} + M$$

holds. Then we have $\rho \le 2^{-\frac{1}{l_0}}$, where ρ is the contraction coefficient of the group G.

Lemma 2. Let $G < \mathrm{Aut}\, T^{(d)}$ be a self-similar contracting group with the contracting coefficient ρ. Then the word problem in *G* is solvable and for any $\varepsilon > 0$ there exists an algorithm of polynomial complexity of degree $\le -\frac{\log d}{\log \rho} + \varepsilon$.

Lemmas formulated above are also proved in [8].

3. Proof of Theorem 1. We are going to give an estimate of the contracting coefficient of the group with help of Lemma 1. Then Lemma 2 will end the proof.

Let *G* be a contracting group. Let *S* be its self-similar finite generating set containing the nucleus. We can add to *S* the set S^{-1} also, so we will consider a symmetric generating set *S*:

$$\begin{cases} g_1 = (g_1^{(1)}, \ldots, g_d^{(1)})\sigma_1, \\ g_2 = (g_1^{(2)}, \ldots, g_d^{(2)})\sigma_2, \\ \qquad \vdots \\ g_n = (g_1^{(n)}, \ldots, g_d^{(n)})\sigma_n, \end{cases}$$

where $g_i^{(j)} \in S$, $\sigma_i \in S_d$.

Since the set *S* is a quasinucleus of a group *G*, all the products $g_i g_j$ of the elements from *S* will contract to *S*. But self-similarity of the set *S* implies that all the restrictions of any product of two elements $g_i g_j$ will be again a product of two elements of *S*. Therefore, some relations like $g_i g_j = g_k$ should be valid in order to have contraction. We suppose that we know the set of those pairs of indexes (*i*,*j*), for which $g_i g_j \in S$.

Let us estimate from above the number of the level at which all the products $g_i g_j$, $g_i, g_j \in S$ contract to *S*. For every product $g_i g_j$, $g_i, g_j \in S$ and every $v \in X^*$ such that $(g_i g_j)|_v \notin S$ the elements $(g_i g_j)|_v$ and $(g_i g_j)|_{vu}$ can not be equal. Otherwise, the product $g_i g_j$ will not contract along the infinite word *vuuu*…. Since the number of the products $g_i g_j \notin S$ is not greater than $n^2 - 1$, all the pairs will contract to *S* not later then at $(n^2 - 1)$-st level.

Let us show that the conditions of Lemma 1 are satisfied for $l_0 = n^2 - 1$ and *M*=1. Let $g = g_{i_1} g_{i_2} \ldots g_{i_{2k+1}}$ be an arbitrary group word in *S* of an odd length *l*(*g*)=2*k*+1. The word *g* can be represented in a form $g_{i_1} g_{i_2} \cdot g_{i_3} g_{i_4} \cdot \ldots \cdot g_{i_{2k+1}} e$, where *e* is the identity automorphism. Then for every word $v = x_1 x_2 \ldots x_{n^2-1} \in X^{n^2-1}$ formulae (2) and (3) imply:

$$g|_v = (g_{i_1} g_{i_2})|_v \cdot (g_{i_3} g_{i_4})|_v \cdot \ldots \cdot (g_{i_{2k+1}} e)|_v .$$

Since every pair $g_{i_j} g_{i_{j+1}}$ contracts to S in the word v, the length of the restriction of g in v $l(g|_v)$ is not greater than $k+1 \le \frac{2k+1}{2}+1=\frac{l(g)}{2}+M$. In the case when G has even length, we get analogically $l(g|_v) \le \frac{l(g)}{2} < \frac{l(g)}{2}+M$. Thus, the conditions of Lemma 1 are satisfied and

$$\rho \le 2^{-\frac{1}{n^2-1}}.$$

By Lemma 2, there exists a polynomial-time algorithm of degree not greater than $-\frac{\log d}{\log \rho}+\varepsilon$ solving the word problem in the group G for any $\varepsilon > 0$. Substituting the last estimate on ρ, we get:

$$-\frac{\log d}{\log \rho}+\varepsilon \le -\frac{\log d}{\log 2^{-\frac{1}{n^2-1}}}+\varepsilon = (n^2-1)\log_2 d+\varepsilon.$$

For example, in the case of a binary tree (i.e., d=2) we have a polynomial algorithm of degree not greater than $n^2-1+\varepsilon$.

References

1. С. В. Алешин. Конечные автоматы и проблема Бернсайда о периодических группах // Матем. заметки. – 1972. – 11. – с.319-328.
1. В. И. Сущанский. Периодические *p*-группы подстановок и неограниченная проблема Бернсайда // ДАН СССР, – 1979. – 247. – 3. – с.557-562.
2. Р. И. Григорчук. К проблеме Бернсайда о периодических группах // Функциональный анализ и приложения. – 1980. – 14. – 1. – с.53-54.
3. N. Gupta and S. Sidki. On the Burnside problem for periodic groups // Math. Z. 1983. – 182. – p.385-388.
4. Р. И. Григорчук. К проблеме Милнора о групповом росте // ДАН СССР, – 1983. – 271. – 1. – с.30-33.
5. R. Grigorchuk. Just infinite branch groups. In A. Shalev, M. P. F. du Sautoi, and D. Segal, editors, New horizons in pro-p groups, volume 184 of Progress in Mathematics, pages 121-179. Birkhauser Verlag, Basel, etc., 2000.
6. В. І. Сущанський. Групи скінченно автоматних підстановок // Доп. НАН України. – 1999. – 2. – с.29-32.
7. V. V. Nekrashevych. Self-similar group actions. – Preprint, Kyiv national Taras Shevchenko University, 2000.

Надійшла до редакції 12.12.2002